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Theory of Non-Homogeneous Anisotropic Elastic Shells Subjected to Arbitrary Temperature Distribution

by

R. E. Miller

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R. E. Miller

ASSISTANT PROFESSOR OF THEORETICAL AND APPLIED MECHANICS

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CONTENTS

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SUMMARY

This paper presents a general theory for small deformations of anisotropic elastic shells subjected to arbitrary temperature distribution.

The shells are assumed to be homogeneous through the thickness except for the coefficient of thermal expansion, but are otherwise unrestricted in homogeneity and isotropy.

The theory is specialized for arbitrary cylinders, circular cones, spheres, and axially symmetrical orthotropic shells of revolution. In addition, two special problems are analyzed.

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INTRODUCTION

The behavior of structural elements subjected to heating has recently assumed importance in many fields. Although most materials now in use are at least approximately isotropic in nature, there are instances where more general analyses based on considerations of anisotropy and non-homogeneity are necessary. Materials often become anisotropic in manufacturing processes such as cold rolling or stretching. In addition, temperature gradients may cause the elastic constants to vary, thus introducing non-homogeneity.

The theory presented in this paper is restricted to small deflections of thin elastic shells. The Kirchhoff assumption, normals to the middle surface remain straight, normal, and inextensional during the deformation, is used throughout. Although they are assumed to be constant through the thickness, the elastic coefficients of the material are arbitrary functions of the surface coordinates x and y of the shell. On the other hand, the thermal coefficients and the temperature distribution are unrestricted.

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NOTATIONS

X, Y, Z = Rectangular coordinates.

$\hat{\imath}, \hat{\jmath}, \hat{k}$ = Unit vectors along the X, Y, Z axes.

$\bar{r}$ = Position vector of the point X, Y, Z.

x, y, z = Shell coordinates based on the lines of principal curvature. See Section I B. The symbols x and y are also used as arbitrary curvilinear coordinates in Section I A.

s = Arc length.

E, F, G = Coefficients in the "first fundamental form" of a surface. See Eqs. (3) and (4).

S = Surface area.

$\hat{n}$ = Unit normal vector to a surface at any point.

e, f, g = Coefficients in the "second fundamental form" of a surface. See Eqs. (8) and (9).

r_1, r_2 = Principal radii of curvature defined by Eq. (11).

M = Mean curvature of a surface defined by Eq. (10).

K = Gaussian curvature of a surface defined by Eq. (10).

A, B = Positive functions of x and y defined by Eq. (13).

α, β, γ = Lamé coefficients for orthogonal curvilinear coordinates. See Eqs. (19) and (20). The symbol α is also used as the apex angle of a cone in Section III B, but no confusion should result.

h = Thickness of the shell.

$\left.\begin{matrix}\sigma_x, \sigma_y, \sigma_z \\ \tau_{xy}, \tau_{yz}, \tau_{zx}\end{matrix}\right\}$ = Stress components at point (x, y, z). See Fig. 1.

$\left.\begin{matrix}N_x, \ldots, N_{yx} \\ Q_x, Q_y \\ M_x, \ldots, M_{yx}\end{matrix}\right\}$ = Tensions, shears, twisting moments, and bending moments defined by Eq. (25).

P_x, P_y, P_z = Components of resultant external force per unit area on middle surface of the shell.

R_x, R_y = Components of the external couple per unit area on the middle surface of the shell.

$\left.\begin{matrix}\epsilon_x, \epsilon_y, \epsilon_z \\ \gamma_{xy}, \gamma_{yz}, \gamma_{zx}\end{matrix}\right\}$ = Strain components at point (x, y, z).

U, V, W = Components of the displacement vector in the directions of the coordinate lines.

u, v, w = Components of the displacement vector of the middle surface in the directions of the coordinate lines.

$\left.\begin{matrix}P_1, Q_1, R_1 \\ P_2, Q_2, R_2 \\ J, K, L, M\end{matrix}\right\}$ = Coefficients defined by Eq. (35).

T = Temperature measured above an arbitrary zero.

b_{ij} = Elastic coefficients. See Eq. (36).

c_i = Thermal coefficients. See Eq. (36).

C_1, C_2 = Constants defined by Eq. (38).

U_o = Strain energy density.

U = Strain energy of the shell.

I = Function defined by Eq. (44).

θ = Shell coordinate used in place of y for the cone and the shell of revolution. See Sections III B and D.

s, c, t = Abbreviations for sin α, cos α, and tan α respectively in the equations of Section III B. The symbols s and c are also used as abbreviations for sin x and ctn x in Section III C.

r = Coordinate used for shell of revolution. See Fig. 5.

V_1, V_2 = Alternative set of displacement components for shell of revolution. See Fig. 6.

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I. GEOMETRICAL PRELIMINARIES

The theory of shells presented here is based upon the differential geometry of surfaces and elasticity theory. The geometrical topics discussed in this chapter are presented in books such as those by Struik[(1)]* or Graustein.[(2)] The following treatment is essentially the same as that presented by Langhaar and Boresi.[(3)]

A. GEOMETRY OF A SURFACE

A point in space may be located with reference to a rectangular cartesian coordinate system (X, Y, Z). The radius vector $\bar{r}$ from the origin to the point may be written as follows:

$$\bar{r} = \hat{\imath}\, X + \hat{\jmath}\, Y + \hat{k}\, Z \tag{1}$$

where $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ are unit vectors directed along the positive X, Y, Z axes respectively.

A surface in space is represented by the equations $X = X(x, y)$, $Y = Y(x, y)$, $Z = Z(x, y)$, where x, y are arbitrary curvilinear coordinates. Then a point on the surface is denoted by the relation $\bar{r} = \bar{r}(x, y)$.

The square of the infinitesimal distance ds between two neighboring points of the surface is

$$ds^2 = d\bar{r} \cdot d\bar{r} = (\bar{r}_x\, dx + \bar{r}_y\, dy)^2 \tag{2}$$

or

$$ds^2 = E\, dx^2 + 2F\, dx\, dy + G\, dy^2. \tag{3}$$

where subscripts x and y denote partial derivatives. By comparison of Eqs. (2) and (3), we obtain

$$\left.\begin{aligned} E &= \bar{r}_x \cdot \bar{r}_x = X_x^2 + Y_x^2 + Z_x^2 \\ F &= \bar{r}_x \cdot \bar{r}_y = X_x X_y + Y_x Y_y + Z_x Z_y \\ G &= \bar{r}_y \cdot \bar{r}_y = X_y^2 + Y_y^2 + Z_y^2. \end{aligned}\right\} \tag{4}$$

Equation (3) is called the "first fundamental form" of the surface.[(1)] Equation (4) shows that E and G are positive. Since ds^2 is positive, Eqs. (3) and (4) yield $EG - F^2 > 0$.

An infinitesimal element of area on the surface is given by the formula

$$dS = \sqrt{EG - F^2}\, dx\, dy. \tag{5}$$

Hence, the area of any part of the surface is given by the relation

$$S = \iint \sqrt{EG - F^2}\, dx\, dy. \tag{6}$$

The unit vector $\hat{n}$ normal to the surface is given by the formula

$$\hat{n} = \frac{\bar{r}_x \times \bar{r}_y}{|\bar{r}_x \times \bar{r}_y|} = \frac{\bar{r}_x \times \bar{r}_y}{\sqrt{EG - F^2}}. \tag{7}$$

This equation fixes the positive sense of $\hat{n}$ as well as its direction.

Another important relation in the theory of surfaces is the following:

$$-d\bar{r} \cdot d\hat{n} = e\, dx^2 + 2f\, dx\, dy + g\, dy^2. \tag{8}$$

Equation (8) is known as the "second fundamental form" of a surface. The coefficients e, f, and g are given by the following formulas:

$$\left.\begin{aligned} e &= \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} X_{xx} & Y_{xx} & Z_{xx} \\ X_x & Y_x & Z_x \\ X_y & Y_y & Z_y \end{vmatrix} \\ f &= \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} X_{xy} & Y_{xy} & Z_{xy} \\ X_x & Y_x & Z_x \\ X_y & Y_y & Z_y \end{vmatrix} \\ g &= \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} X_{yy} & Y_{yy} & Z_{yy} \\ X_x & Y_x & Z_x \\ X_y & Y_y & Z_y \end{vmatrix} \end{aligned}\right\} \tag{9}$$

The extreme values of the curvatures of normal sections of the surface at a given point, called the "principal curvatures" of the surface, are denoted by $1/r_1$ and $1/r_2$. The mean curvature M and the Gaussian curvature K are defined as follows:

$$M = \frac{1}{2}(1/r_1 + 1/r_2), \quad K = 1/(r_1 r_2). \tag{10}$$

In the following, it is assumed that lines of principal curvature coincide with coordinate lines x, y. With this restriction, $f = F = 0$. Then the principal curvatures are determined by the following formulas:

* Superscripts in parentheses refer to references listed in the Bibliography.

$$1/r_1 = e/E, \quad 1/r_2 = g/G. \tag{11}$$

Equation (11) determines the signs of r_1 and r_2 as well as their magnitudes.

A relation of differential geometry which finds application in shell theory is Rodrigues' theorem.[1] For $f = F = 0$, it may be expressed in the form

$$\frac{\partial \bar{n}}{\partial x} = -\frac{1}{r_1}\frac{\partial \bar{r}}{\partial x}, \quad \frac{\partial \bar{n}}{\partial y} = -\frac{1}{r_2}\frac{\partial \bar{r}}{\partial y}. \tag{12}$$

For orthogonal coordinates, it is convenient to introduce the notations

$$A^2 = E, \quad B^2 = G. \tag{13}$$

It is shown in differential geometry that the functions E, G, e, and g satisfy three differential equations of compatibility, known as the Gauss-Codazzi equations. For orthogonal surface coordinates ($F = 0$), the Gauss equation is

$$-KAB = \frac{\partial}{\partial x}\left(\frac{B_x}{A}\right) + \frac{\partial}{\partial y}\left(\frac{A_y}{B}\right), \tag{14}$$

where K is the Gaussian curvature. The Gaussian curvature is a bending invariant. For $f = F = 0$ (that is, for coordinate lines which coincide with lines of principal curvature), the Codazzi equations are

$$\frac{\partial}{\partial y}\left(\frac{A}{r_1}\right) = \frac{1}{r_2}\frac{\partial A}{\partial y}, \quad \frac{\partial}{\partial x}\left(\frac{B}{r_2}\right) = \frac{1}{r_1}\frac{\partial B}{\partial x}. \tag{15}$$

B. GEOMETRY OF A SHELL

It was noted in Section A that $\bar{r} = \bar{r}(x, y)$ represents a surface in space. In addition to coordinates x, y, a third coordinate z is required to represent a shell. Thus, a point in a shell may be represented by the relation $\bar{r} = \bar{r}(x, y, z)$. If coordinates x, y, z are orthogonal at every point, they are called orthogonal curvilinear coordinates. Then, since the vector derivatives $\bar{r}_x$, $\bar{r}_y$, $\bar{r}_z$ are tangent to their respective coordinate lines,

$$\bar{r}_x \cdot \bar{r}_y = \bar{r}_y \cdot \bar{r}_z = \bar{r}_z \cdot \bar{r}_x = 0. \tag{16}$$

The square of the infinitesimal distance between two neighboring points is

$$ds^2 = d\bar{r} \cdot d\bar{r} = (\bar{r}_x\, dx + \bar{r}_y\, dy + \bar{r}_z\, dz)^2. \tag{17}$$

With Eq. (16), Eq. (17) may be written simply

$$ds^2 = \bar{r}_x \cdot \bar{r}_x\, dx^2 + \bar{r}_y \cdot \bar{r}_y\, dy^2 + \bar{r}_z \cdot \bar{r}_z\, dz^2. \tag{18}$$

Let the following notations be introduced:

$$\alpha^2 = \bar{r}_x \cdot \bar{r}_x, \quad \beta^2 = \bar{r}_y \cdot \bar{r}_y, \quad \gamma^2 = \bar{r}_z \cdot \bar{r}_z. \tag{19}$$

With this notation, Eq. (18) becomes

$$ds^2 = \alpha^2\, dx^2 + \beta^2\, dy^2 + \gamma^2\, dz^2. \tag{20}$$

The coefficients α, β, γ are functions of x, y, z. They are called "Lamé coefficients."

A special type of orthogonal curvilinear coordinates is employed in shell theory. Let x and y denote surface coordinates on the middle surface of the shell. Let z denote the distance of a point measured from the middle surface in the direction of the normal to the middle surface. Positive z is measured in the positive sense of the unit normal $\bar{n}$ of the middle surface (see Eq. 7). The lateral boundary of the shell is $z = \pm h/2$, where h is the thickness of the shell. In general, h is a function of x and y. When x, y, and z are defined in this way, they are called shell coordinates.

If the shell coordinates are orthogonal, the coordinate lines on the middle surface coincide with the lines of principal curvature. This follows from a theorem of Dupin which states: *The surfaces of a triply orthogonal system intersect on their lines of principal curvature.* For orthogonal shell coordinates, the square of the infinitesimal distance ds between two neighboring points of the middle surface is

$$ds^2 = A^2\, dx^2 + B^2\, dy^2. \tag{21}$$

For orthogonal shell coordinates, the Lamé coefficients (Eq. 19) become

$$\alpha = A(1 - z/r_1), \quad \beta = B(1 - z/r_2), \quad \gamma = 1, \tag{22}$$

where r_1 and r_2 are the principal radii of curvature of the middle surface (see Eq. 11).

II. GENERAL ANISOTROPIC SHELLS

A. EQUILIBRIUM RELATIONS

The stress notations used are shown in Fig. 1. The stress σ_x is normal to a plane which is perpendicular to the x-axis, and the stresses τ_{xy}, τ_{xz} are tangent to this plane and directed in the y and z directions, respectively. It is shown in the theory of elasticity[4] that the shearing stresses obey the following relations:

$$\tau_{xy} = \tau_{yx}, \qquad \tau_{xz} = \tau_{zx}, \qquad \tau_{yz} = \tau_{zy}. \tag{23}$$

Love[4] has derived the differential equations of equilibrium for any orthogonal coordinates. For shell coordinates $(\gamma = 1)$ in the absence of body forces Love's equations are

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}(\beta\sigma_x) + \frac{\partial}{\partial y}(\alpha\tau_{xy}) + \frac{\partial}{\partial z}(\alpha\beta\tau_{xz}) \\
&\qquad + \frac{\partial\alpha}{\partial y}\tau_{xy} + \beta\frac{\partial\alpha}{\partial z}\tau_{xz} - \frac{\partial\beta}{\partial x}\sigma_y = 0 \\
&\frac{\partial}{\partial x}(\beta\tau_{xy}) + \frac{\partial}{\partial y}(\alpha\sigma_y) + \frac{\partial}{\partial z}(\alpha\beta\tau_{yz}) \\
&\qquad + \frac{\partial\beta}{\partial x}\tau_{xy} + \alpha\frac{\partial\beta}{\partial z}\tau_{yz} - \frac{\partial\alpha}{\partial y}\sigma_x = 0 \\
&\frac{\partial}{\partial x}(\beta\tau_{xz}) + \frac{\partial}{\partial y}(\alpha\tau_{yz}) + \frac{\partial}{\partial z}(\alpha\beta\sigma_z) \\
&\qquad - \beta\frac{\partial\alpha}{\partial z}\sigma_x - \alpha\frac{\partial\beta}{\partial z}\sigma_y = 0
\end{aligned}\right\} \tag{24}$$

These equations are exact only if x, y, z are orthogonal shell coordinates for the *deformed* shell. For small displacements, the effects of deformation on the equilibrium relations are usually neglected.

The tensions N_x, N_y, shears N_{xy}, N_{yx}, Q_x, Q_y, twisting moments M_{xy}, M_{yx}, and bending moments M_x, M_y per unit length of the middle surface may be expressed in terms of the stress components σ_x, σ_y, τ_{xy}, τ_{xz}, τ_{yz} by considering an infinitesimal element of the shell (Fig. 2). The positive senses of N_x, N_y, N_{xy}, N_{yx}, Q_x, Q_y are indicated in Fig. 2, where double-headed arrows denote moments with positive sense given by the right-hand-rule convention.

In Fig. 2, the total tensile force on the differential element in the x direction is $N_x B\,dy$, where N_x is the tension per unit length on the plane perpendicular to the x-axis. Also, with the notation of Fig. 1, the total tensile force may be written in the form

$$\int \sigma_x \beta\,dy\,dz = dy \int \beta\,\sigma_x\,dz.$$

Hence,

$$N_x = \frac{1}{B}\int \beta\,\sigma_x\,dz = \int_{-h/2}^{h/2} \sigma_x(1 - z/r_2)\,dz.$$

Similarly, N_y, N_{xy}, . . . , Q_y may be expressed. Thus, we obtain the following equations:

$$N_x = \int_{-h/2}^{h/2} \sigma_x(1 - z/r_2)\,dz, \tag{25}$$

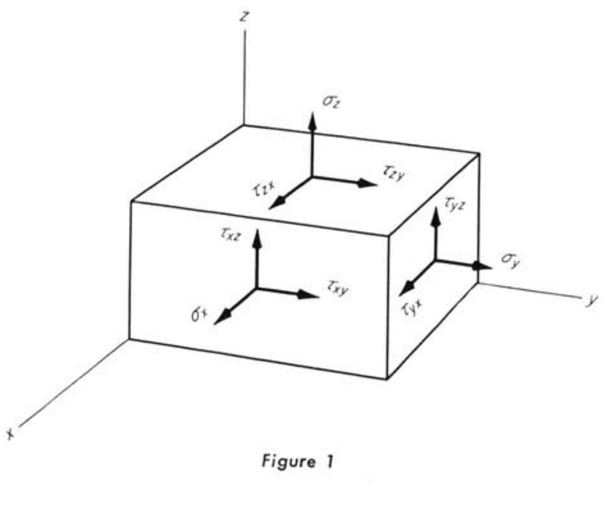

Figure 1

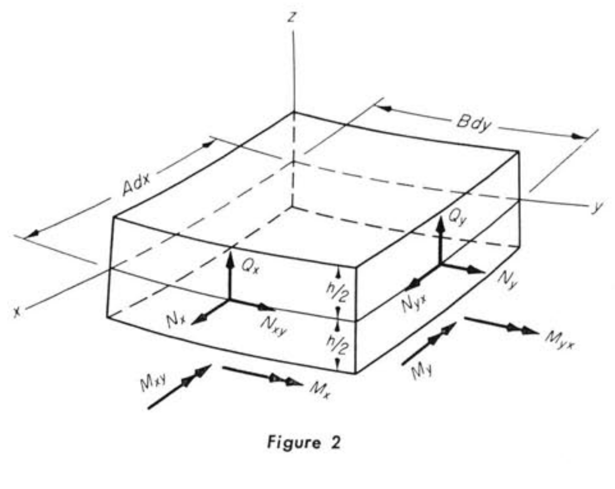

Figure 2

$$
\left.\begin{aligned}
N_y &= \int_{-h/2}^{h/2} \sigma_y (1 - z/r_1)\, dz, \\
N_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} (1 - z/r_2)\, dz, \\
N_{yx} &= \int_{-h/2}^{h/2} \tau_{xy} (1 - z/r_1)\, dz, \\
M_x &= \int_{-h/2}^{h/2} \sigma_x z (1 - z/r_2)\, dz, \\
M_y &= \int_{-h/2}^{h/2} \sigma_y z (1 - z/r_1)\, dz, \\
M_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} z (1 - z/r_2)\, dz, \\
M_{yx} &= \int_{-h/2}^{h/2} \tau_{xy} z (1 - z/r_1)\, dz, \\
Q_x &= \int_{-h/2}^{h/2} \tau_{xz} (1 - z/r_2)\, dz, \\
Q_y &= \int_{-h/2}^{h/2} \tau_{yz} (1 - z/r_1)\, dz.
\end{aligned}\right\} \tag{25}
$$

These representations of tensions, shears, and moments are similar to those used by Flügge.[5]

The equilibrium equations for a shell are derived by Langhaar and Boresi.[3] The results are repeated here.

$$
\begin{aligned}
\frac{\partial}{\partial x}(BN_x) + \frac{\partial}{\partial y}(AN_{yx}) + N_{xy}\frac{\partial A}{\partial y} - N_y \frac{\partial B}{\partial x} \\
- \frac{ABQ_x}{r_1} + ABP_x = 0
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x}(BN_{xy}) + \frac{\partial}{\partial y}(AN_y) + N_{yx}\frac{\partial B}{\partial x} - N_x \frac{\partial A}{\partial y} \\
- \frac{ABQ_y}{r_2} + ABP_y = 0
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x}(BQ_x) + \frac{\partial}{\partial y}(AQ_y) + \frac{AB}{r_1} N_x \\
+ \frac{AB}{r_2} N_y + ABP_z = 0
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x}(BM_x) + \frac{\partial}{\partial y}(AM_{yx}) + M_{xy}\frac{\partial A}{\partial y} \\
- M_y \frac{\partial B}{\partial x} - ABQ_x + ABR_y = 0
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x}(BM_{xy}) + \frac{\partial}{\partial y}(AM_y) + M_{yx}\frac{\partial B}{\partial x} \\
- M_x \frac{\partial A}{\partial y} - ABQ_y - ABR_x = 0
\end{aligned} \tag{30}
$$

$$
\frac{M_{yx}}{r_2} - \frac{M_{xy}}{r_1} = N_{yx} - N_{xy}. \tag{31}
$$

In these equations P_x, P_y, P_z denote the components of the resultant external force per unit area on the middle surface, and R_x and R_y denote components of the external couple on the middle surface of the shell. If R_x and R_y are neglected, the moment equilibrium equation is

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left\{\frac{1}{A}\left[\frac{\partial}{\partial x}(BM_x) + \frac{\partial}{\partial y}(AM_{yx}) \right.\right. \\
&\qquad \left.\left. + M_{xy}\frac{\partial A}{\partial y} - M_y\frac{\partial B}{\partial x}\right]\right\} \\
&\quad + \frac{\partial}{\partial y}\left\{\frac{1}{B}\left[\frac{\partial}{\partial x}(BM_{xy}) + \frac{\partial}{\partial y}(AM_y) \right.\right. \\
&\qquad \left.\left. + M_{yx}\frac{\partial B}{\partial x} - M_x\frac{\partial A}{\partial y}\right]\right\} \\
&\quad + \frac{AB}{r_1} N_x + \frac{AB}{r_2} N_y + ABP_z = 0.
\end{aligned} \tag{32}
$$

B. STRAIN-DISPLACEMENT RELATIONS

Let U, V, W denote the components of the displacement vector in the directions of the coordinate lines. The general expressions for the strains ϵ_x, ϵ_y, ϵ_z, γ_{xy}, γ_{yz}, γ_{zx}, in terms of U, V, W, have been derived by Novozhilov[6] and Shaw.[7] However, the objective of the theory of shells is to reduce the problem of three-dimensional elasticity to two dimensions. To attain this goal, we employ the Kirchhoff assumption, which states that under a deformation, normals to the middle surface remain straight, normal, and inextensional. This implies that $\epsilon_z = \gamma_{yz} = \gamma_{zx} = 0$. It follows that the displacement vector U, V, W is a linear function of the thickness coordinate z. Also, for small deflections, quadratic terms in U, V, W are usually neglected.

With these approximations, the general formulas for the strain components ϵ_x, ϵ_y, γ_{xy} at any point in the shell in terms of the displacement components u, v, w of the middle surface have been derived by Love[4] and by Langhaar and Boresi.[3] The results[3] are

$$
\left.\begin{aligned}
\epsilon_x &= \frac{u_x}{A} + (1 - z/r_1)^{-1}\left(\frac{vA_y}{AB} - \frac{w}{r_1}\right) \\
&\quad - z(1 - z/r_1)^{-1}\left[\frac{u}{A}\frac{\partial}{\partial x}\left(\frac{1}{r_1}\right)\right. \\
&\quad \left. + \frac{1}{A}\frac{\partial}{\partial x}\left(\frac{w_x}{A}\right) + \frac{vA_y}{ABr_2} + \frac{A_y w_y}{AB^2}\right], \\
\epsilon_y &= \frac{v_y}{B} + (1 - z/r_2)^{-1}\left(\frac{uB_x}{AB} - \frac{w}{r_2}\right)
\end{aligned}\right| \tag{33}
$$

$$
\left.\begin{aligned}
&- z\,(1 - z/r_2)^{-1}\left[\frac{v}{B}\,\frac{\partial}{\partial y}\left(\frac{1}{r_2}\right)\right.\\
&\left.+ \frac{1}{B}\,\frac{\partial}{\partial y}\left(\frac{w_y}{B}\right) + \frac{uB_x}{ABr_1} + \frac{B_x w_x}{A^2B}\right],\\
\gamma_{xy} &= (1 - z/r_1)^{-1}\,\frac{B}{A}\,\frac{\partial}{\partial x}\left(\frac{v}{B}\right)\\
&+ (1 - z/r_2)^{-1}\,\frac{A}{B}\,\frac{\partial}{\partial y}\left(\frac{u}{A}\right)\\
&- z\,(1 - z/r_1)^{-1}\,\frac{B}{Ar_2}\,\frac{\partial}{\partial x}\left(\frac{v}{B}\right)\\
&+ \frac{1}{A}\,\frac{\partial}{\partial x}\left(\frac{w_y}{B}\right) - \frac{A_y w_x}{A^2B}\\
&- z\,(1 - z/r_2)^{-1}\,\frac{A}{Br_1}\,\frac{\partial}{\partial y}\left(\frac{u}{A}\right)\\
&+ \frac{1}{B}\,\frac{\partial}{\partial y}\left(\frac{w_x}{A}\right) - \frac{B_x w_y}{AB^2}.
\end{aligned}\right\} \qquad (33)
$$

For the purposes of this paper, it is convenient to rewrite Eq. (33) in the following form:

$$
\left.\begin{aligned}
\epsilon_x &= P_1 + \frac{Q_1}{1 - z/r_1} - \frac{zR_1}{1 - z/r_1},\\
\epsilon_y &= P_2 + \frac{Q_2}{1 - z/r_2} - \frac{zR_2}{1 - z/r_2},\\
\gamma_{xy} &= \frac{J}{1 - z/r_1} + \frac{K}{1 - z/r_2}\\
&\quad - \frac{zL}{1 - z/r_1} - \frac{zM}{1 - z/r_2},
\end{aligned}\right\} \qquad (34)
$$

where

$$
\left.\begin{aligned}
P_1 &= \frac{u_x}{A}, \quad P_2 = \frac{v_y}{B},\\
Q_1 &= \frac{vA_y}{AB} - \frac{w}{r_1}, \qquad Q_2 = \frac{uB_x}{AB} - \frac{w}{r_2},\\
R_1 &= \frac{u}{A}\,\frac{\partial}{\partial x}\left(\frac{1}{r_1}\right) + \frac{1}{A}\,\frac{\partial}{\partial x}\left(\frac{w_x}{A}\right)\\
&\quad + \frac{vA_y}{ABr_2} + \frac{A_y w_y}{AB^2},\\
J &= \frac{B}{A}\,\frac{\partial}{\partial x}\left(\frac{v}{B}\right),\\
R_2 &= \frac{v}{B}\,\frac{\partial}{\partial y}\left(\frac{1}{r_2}\right) + \frac{1}{B}\,\frac{\partial}{\partial y}\left(\frac{w_y}{B}\right)\\
&\quad + \frac{uB_x}{ABr_1} + \frac{B_x w_x}{A^2B},\\
K &= \frac{A}{B}\,\frac{\partial}{\partial y}\left(\frac{u}{A}\right),\\
L &= \frac{B}{Ar_2}\,\frac{\partial}{\partial x}\left(\frac{v}{B}\right)\\
&\quad + \frac{1}{A}\,\frac{\partial}{\partial x}\left(\frac{w_y}{B}\right) - \frac{A_y w_x}{A^2B},\\
M &= \frac{A}{Br_1}\,\frac{\partial}{\partial y}\left(\frac{u}{A}\right)\\
&\quad + \frac{1}{B}\,\frac{\partial}{\partial y}\left(\frac{w_x}{A}\right) - \frac{B_x w_y}{AB^2}.
\end{aligned}\right\} \qquad (35)
$$

C. STRESS-STRAIN-TEMPERATURE RELATIONS

For linearly elastic anisotropic material, the plane stress-strain-temperature relations for a shell are

$$\sigma_i = b_{ij}\,\epsilon_j - c_i\,T \qquad (36)$$

where $b_{ij} = b_{ji}$ are elastic constants, and c_i are thermal coefficients.[4] A repeated index is to be summed from one to three. In Eq. (36), the following notations have been used:

$$\sigma_1 = \sigma_x, \qquad \sigma_2 = \sigma_y, \qquad \sigma_3 = \tau_{xy},$$

$$\epsilon_1 = \epsilon_x, \qquad \epsilon_2 = \epsilon_y, \qquad \epsilon_3 = \gamma_{xy}.$$

To obtain tensions, shears, and moments in terms of u, v, w, we substitute Eq. (34) into Eq. (36) and then substitute the result into Eq. (25). Thus, we obtain

$$
\left.\begin{aligned}
N_x &= h\,\{b_{11}\,[P_1 + Q_1\,(1 - C_1) + r_1C_1R_1]\\
&\quad + b_{12}\,[P_2 + Q_2] + b_{13}\,[K + J\,(1 - C_1)\\
&\quad + Lr_1C_1]\} - \int_{-h/2}^{h/2} c_1T\,(1 - z/r_2)\,dz,\\
N_y &= h\,\{b_{22}\,[P_2 + Q_2\,(1 - C_2) + r_2C_2R_2]\\
&\quad + b_{12}\,[P_1 + Q_1] + b_{23}\,[J + K\,(1 - C_2)\\
&\quad + Mr_2C_2]\} - \int_{-h/2}^{h/2} c_2T\,(1 - z/r_1)\,dz,\\
N_{xy} &= h\,\{b_{13}\,[P_1 + Q_1\,(1 - C_1) + r_1C_1R_1]\\
&\quad + b_{23}\,[P_2 + Q_2] + b_{33}\,[K + J\,(1 - C_1)\\
&\quad + Lr_1C_1]\} - \int_{-h/2}^{h/2} c_3T\,(1 - z/r_2)\,dz,\\
N_{yx} &= h\,\{b_{23}\,[P_2 + Q_2\,(1 - C_2) + r_2C_2R_2]\\
&\quad + b_{13}\,[P_1 + Q_1] + b_{33}\,[J + K\,(1 - C_2)\\
&\quad + Mr_2C_2]\} - \int_{-h/2}^{h/2} c_3T\,(1 - z/r_1)\,dz,\\
M_x &= \frac{-h^3}{12r_2}\left\{b_{11}\left[P_1 + \frac{12}{h^2}\,Q_1C_1r_1r_2\right.\right.
\end{aligned}\right\} \qquad (37)
$$

$$
\begin{aligned}
&+ R_1 r_1\left(1 - \frac{12 r_1 r_2 C_1}{h^2}\right)\Big] \\
&+ b_{12}\left[P_2 + R_2 r_2\right] + b_{13}\left[\frac{12}{h^2} J C_1 r_1 r_2\right. \\
&\left. + M r_2 + L r_1\left(1 - \frac{12 r_1 r_2 C_1}{h^2}\right)\right]\Big\} \\
&- \int_{-h/2}^{h/2} c_1 T\, z\,(1 - z/r_2)\, dz, \\
M_y = \frac{-h^3}{12 r_1}&\Big\{b_{22}\Big[P_2 + \frac{12}{h^2} Q_2 C_2 r_1 r_2 \\
&+ R_2 r_2\left(1 - \frac{12 r_1 r_2 C_2}{h^2}\right)\Big] \\
&+ b_{12}\left[P_1 + R_1 r_1\right] + b_{23}\left[\frac{12}{h^2} K C_2 r_1 r_2\right. \\
&\left. + L r_1 + M r_2\left(1 - \frac{12 r_1 r_2 C_2}{h^2}\right)\right]\Big\} \\
&- \int_{-h/2}^{h/2} c_2 T\, z\,(1 - z/r_1)\, dz, \\
M_{xy} = \frac{-h^3}{12 r_2}&\Big\{b_{31}\Big[P_1 + \frac{12}{h^2} Q_1 C_1 r_1 r_2 \\
&+ R_1 r_1\left(1 - \frac{12 r_1 r_2 C_1}{h^2}\right)\Big] \\
&+ b_{32}\left[P_2 + R_2 r_2\right] + b_{33}\left[\frac{12}{h^2} J C_1 r_1 r_2\right. \\
&\left. + M r_2 + L r_1\left(1 - \frac{12 r_1 r_2 C_1}{h^2}\right)\right]\Big\} \\
&- \int_{-h/2}^{h/2} c_3 T\, z\,(1 - z/r_2)\, dz, \\
M_{yx} = \frac{-h^3}{12 r_1}&\Big\{b_{32}\Big[P_2 + \frac{12}{h^2} Q_2 C_2 r_1 r_2 \\
&+ R_2 r_2\left(1 - \frac{12 r_1 r_2 C_2}{h^2}\right)\Big] \\
&+ b_{13}\left[P_1 + R_1 r_1\right] + b_{33}\left[\frac{12}{h^2} K C_2 r_1 r_2\right. \\
&\left. + L r_1 + M r_2\left(1 - \frac{12 r_1 r_2 C_2}{h^2}\right)\right]\Big\} \\
&- \int_{-h/2}^{h/2} c_3 T\, z\,(1 - z/r_1)\, dz,
\end{aligned}
\qquad (37)
$$

where

$$
\left.
\begin{aligned}
C_1 &= (1 - r_1/r_2)\left[1 - \frac{r_1}{h}\, ln\left(\frac{1 + h/2\, r_1}{1 - h/2\, r_1}\right)\right], \\
C_2 &= (1 - r_2/r_1)\left[1 - \frac{r_2}{h}\, ln\left(\frac{1 + h/2\, r_2}{1 - h/2\, r_2}\right)\right].
\end{aligned}
\right\} \qquad (38)
$$

D. STRAIN ENERGY OF THE SHELL

To obtain solutions of shell problems by energy methods, the strain energy of the shell in terms of the displacement components of the middle surface is required. The sum of the strain energy integral and the potential energy of the external loads equals the total potential energy of the system. By minimization of the total potential energy, expressions for u, v, w are obtained. When u, v, w are known, strain components, stress components, and tractions may be computed by Eqs. (34), (36), and (37). Since exact solutions for the displacement components u, v, and w are often difficult to obtain, approximate methods, such as the Rayleigh-Ritz procedure,[8] are frequently employed.

The expression for the strain energy of the shell is obtained by integrating the strain energy density throughout the volume. The stress components are related to the strain energy density[4] by the relations

$$\sigma_i = \frac{\partial U_o}{\partial \epsilon_i}. \qquad (39)$$

Substituting Eq. (36) into Eq. (39) and integrating, we obtain

$$U_o = \frac{1}{2} b_{ij}\epsilon_i\epsilon_j - c_i\epsilon_i T + C(T). \qquad (40)$$

Since the arbitrary function $C(T)$ is immaterial in the application of energy methods, it is disregarded. Thus we obtain

$$U_o = \frac{1}{2} b_{ij}\,\epsilon_i\epsilon_j - c_i\epsilon_i T. \qquad (41)$$

The total strain energy is

$$U = \iiint U_o\, \alpha\beta\, dx\, dy\, dz. \qquad (42)$$

Integration with respect to z may be performed readily with the assumption that b_{ij} are independent of z. Thus, substituting Eq. (34) into Eq. (41), substituting the results into Eq. (42), and integrating through the thickness, we obtain

$$
\begin{aligned}
U = \iint h \Big\{ &\frac{b_{11}}{2}(P_1 + Q_1)^2 + \frac{b_{22}}{2}(P_2 + Q_2)^2 \\
&+ b_{12}(P_1 + Q_1)(P_2 + Q_2) \\
&+ \frac{b_{33}}{2}(K + J)^2 + b_{13}(P_1 + Q_1)(J + K) \\
&+ b_{23}(P_2 + Q_2)(J + K)\Big\} AB\, dx\, dy
\end{aligned}
$$

$$+\iint \frac{h^3}{12}\left\{\frac{b_{11}}{2}\left[\frac{(P_1+r_1R_1)^2}{r_1r_2}\right.\right.$$

$$\left.-\frac{12C_1}{h^2}(Q_1-r_1R_1)^2\right]+\frac{b_{22}}{2}\left[\frac{(P_2+r_2R_2)^2}{r_1r_2}\right.$$

$$\left.-\frac{12C_2}{h^2}(Q_2-r_2R_2)^2\right]+\frac{b_{33}}{2}\left[\frac{(r_2M+r_1L)^2}{r_1r_2}\right.$$

$$\left.-\frac{12C_1}{h^2}(J-r_1L)^2-\frac{12C_2}{h^2}(K-r_2M)^2\right]$$

$$+b_{12}\frac{(P_1+r_1R_1)(P_2+r_2R_2)}{r_1r_2}$$

$$+b_{13}\left[\frac{(r_1L+r_2M)(P_1+r_1R_1)}{r_1r_2}\right.$$

$$\left.-\frac{12C_1}{h^2}(J-r_1L)(Q_1-r_1R_1)\right]$$

$$+b_{23}\left[\frac{(r_2M+r_1L)(P_2+r_2R_2)}{r_1r_2}\right.$$

$$\left.\left.-\frac{12C_2}{h^2}(K-r_2M)(Q_2-r_2R_2)\right]\right\}AB\,dx\,dy$$

$$+\iint I\,AB\,dx\,dy \tag{43}$$

where

$$I = -\int_{-h/2}^{h/2}[(1-z/r_1)(1-z/r_2)(c_1P_1+c_2P_2)$$

$$+(1-z/r_1)(c_3K+c_2Q_2)$$

$$+(1-z/r_2)(c_3J+c_1Q_1)$$

$$-z(1-z/r_1)(c_2R_2+c_3M)$$

$$-z(1-z/r_2)(c_3L+c_1R_1)]\,T\,dz. \tag{44}$$

Equation (43) is the general expression for the strain energy of an anisotropic, non-homogeneous, elastic shell with arbitrary temperature distribution. It is restricted to small deflections. For an isotropic shell, Eq. (43) reduces to the result obtained by Langhaar and Boresi.[3]

III. SPECIALIZATIONS OF THE GENERAL THEORY

In this chapter, the equations developed in Chapter II are specialized for several common types of shells.

A. CYLINDRICAL SHELL

In this section, equations are presented for an arbitrary cylindrical shell oriented with its generators parallel to the Z-axis. Then $X = X(y)$, $Y = Y(y)$, $Z = x$, where x is distance measured along a generator and y is arc length around the girth of the cylinder.

By Eqs. (4), (9), (11), and (13),

$$\left.\begin{aligned} &E = 1, \quad F = 0, \quad G = 1, \quad A = 1, \\ &B = 1, \quad 1/r_1 = 0, \quad r_2 = r(y). \end{aligned}\right\} \tag{45}$$

With Eq. (45), Eqs. (35) and (38) become

$$\left.\begin{aligned} &P_1 = u_x, \quad P_2 = v_y, \quad Q_1 = 0, \quad Q_2 = -w/r, \\ &R_1 = w_{xx}, \quad R_2 = w_{yy} - vr_y/r^2, \quad J = v_x, \\ &K = u_y, \quad L = v_x/r + w_{xy}, \quad M = w_{xy}, \\ &C_1 = 0, \quad C_2 = 1 - \frac{r}{h}\ln\left(\frac{1 + h/2r}{1 - h/2r}\right) = C. \end{aligned}\right\} \tag{46}$$

Substitution of Eq. (46) into Eq. (37) yields formulas for tensions, shears, and moments:

$$\left.\begin{aligned}
N_x &= h\left\{b_{11}\left[u_x + \frac{h^2 w_{xx}}{12r}\right] + b_{12}\left[v_y - w/r\right] + b_{13}\left[u_y + v_x + \frac{h^2}{12r}\left(v_x/r + w_{xy}\right)\right]\right\} - \int_{-h/2}^{h/2} c_1 T\,(1 - z/r)\,dz, \\
N_y &= h\left\{b_{22}\left[v_y - w/r + C(w/r + rw_{yy} - vr_y/r)\right] + b_{12}\,u_x + b_{23}\left[v_x + u_y + C(rw_{xy} - u_y)\right]\right\} - \int_{-h/2}^{h/2} c_2 T\,dz, \\
N_{xy} &= h\left\{b_{33}\left[u_y + v_x + h^2\left(v_x + rw_{xy}\right)/12r^2\right] + b_{13}\left[u_x + h^2 w_{xx}/12r\right] + b_{23}\left[v_y - w/r\right]\right\} - \int_{-h/2}^{h/2} c_3 T\,(1 - z/r)\,dz, \\
N_{yx} &= h\left\{b_{33}\left[v_x + u_y + C\left(rw_{xy} - u_y\right)\right] + b_{13}\,u_x + b_{23}\left[v_y - w/r + C(w/r + rw_{yy} - vr_y/r)\right]\right\} - \int_{-h/2}^{h/2} c_3 T\,dz, \\
M_x &= -\frac{h^3}{12r}\left\{b_{11}\left[u_x + rw_{xx}\right] + b_{12}\left[v_y + rw_{yy} - vr_y/r\right] + 2b_{13}\left[v_x + rw_{xy}\right]\right\} - \int_{-h/2}^{h/2} c_1 Tz\,(1 - z/r)\,dz, \\
M_y &= -\frac{h^3}{12}\left\{b_{22}\left(12C/h^2\right)\left(vr_y - r^2 w_{yy} - w\right) + b_{12} w_{xx} + b_{23}\left[\frac{12C}{h^2}\left(ru_y - r^2 w_{xy}\right) + \frac{v_x}{r} + w_{xy}\right]\right\} - \int_{-h/2}^{h/2} c_2 Tz\,dz, \\
M_{xy} &= -\frac{h^3}{12r}\left\{2b_{33}\left[v_x + rw_{xy}\right] + b_{13}\left[u_x + rw_{xx}\right] + b_{23}\left[v_y + rw_{yy} - vr_y/r\right]\right\} - \int_{-h/2}^{h/2} c_3 Tz\,(1 - z/r)\,dz, \\
M_{yx} &= -\frac{h^3}{12}\left\{b_{33}\left[v_x/r + w_{xy} + \left(12Cr/h^2\right)\left(u_y - rw_{xy}\right)\right] + b_{13}\,w_{xx} + b_{23}\left(12C/h^2\right)\left(v_y - r^2 w_{yy} - w + vr_y\right)\right\} - \int_{-h/2}^{h/2} c_3 Tz\,dz.
\end{aligned}\right\} \tag{47}$$

By Eqs. (26) to (31) and Eq. (45), the equilibrium equations are

$$\left.\begin{aligned}
&\frac{\partial N_x}{\partial x}+\frac{\partial N_{yx}}{\partial y}+P_x=0\\
&\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}-\frac{Q_y}{r}+P_y=0\\
&\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+\frac{N_y}{r}+P_z=0\\
&\frac{\partial M_x}{\partial x}+\frac{\partial M_{yx}}{\partial y}-Q_x+R_y=0\\
&\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y-R_x=0\\
&N_{yx}-N_{xy}=\frac{M_{yx}}{r}.
\end{aligned}\right\}\qquad(48)$$

Also, Eqs. (32) and (45) yield

$$\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_{yx}}{\partial x\partial y}+\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+\frac{N_y}{r}+P_z=0.\qquad(49)$$

The strain components, by Eqs. (34) and (46), are

$$\left.\begin{aligned}
\epsilon_x&=u_x-zw_{xx},\\
\epsilon_y&=v_y-w/(r-z)-\frac{zr}{r-z}\,(w_{yy}-vr_y/r^2),\\
\gamma_{xy}&=v_x+ru_y/(r-z)-z\,(v_x/r+w_{xy})\\
&\quad-z\left(\frac{r}{r-z}\right)w_{xy}.
\end{aligned}\right\}\qquad(50)$$

By Eqs. (43) and (46), the strain energy for the cylindrical shell is

$$\begin{aligned}
U=&\iint\frac{1}{2}\,h\,\{b_{11}\,u_x^2+b_{22}\,(v_y-w/r)^2+b_{33}\,(v_x\\
&+u_y)^2+2b_{12}u_x\,(v_y-w/r)+2b_{13}u_x\,(v_x\\
&+u_y)+2b_{23}\,(v_y-w/r)\,(u_y+v_x)\}\,dx\,dy\\
&+\iint\frac{h^3}{24}\,\{b_{11}\,(w_{xx}^2+2u_xw_{xx}/r)\\
&-b_{22}\,(12C/h^2)\,(w/r+rw_{yy}-vr_y/r)^2\\
&+b_{33}\,[3\,(v_x/r+w_{xy})^2-(12C/h^2)\,(u_y\\
&-rw_{xy})^2+2b_{12}\,[w_{xx}v_y/r+w_{xx}\,(w_{yy}\\
&-vr_y/r^2)]+2b_{13}\,[(u_x/r)\,(v_x/r+w_{xy})\\
&+2w_{xx}\,(v_x/r+w_{xy})]+2b_{23}\,[(v_x/r\\
&+w_{xy})\,(v_y/r+w_{yy}-vr_y/r^2)\\
&+(12C/h^2)\,(u_y-rw_{xy})\,(rw_{yy}+w/r\\
&-vr_y/r)]\}\,dx\,dy+\iint I\,dx\,dy,
\end{aligned}\qquad(51)$$

where

$$\begin{aligned}
I=&-\int_{-h/2}^{h/2}[(1-z/r)\,(c_1u_x+c_2v_y+c_3v_x)\\
&+c_3u_y-c_2w/r-z\,(1-z/r)\,(c_3w_{xy}\\
&+c_3v_x/r+c_1w_{xx})-z\,(c_2w_{yy}+c_3w_{xy}\\
&-c_2vr_y/r^2)]\,T\,dz.
\end{aligned}\qquad(52)$$

B. CONICAL SHELL OF CIRCULAR CROSS SECTION

The middle surface of the shell is given by the equations $X=x\cos\alpha$, $Y=x\sin\alpha\cos\theta$, $Z=x\sin\alpha\sin\theta$, where x, θ are shell coordinates indicated in Fig. 3.

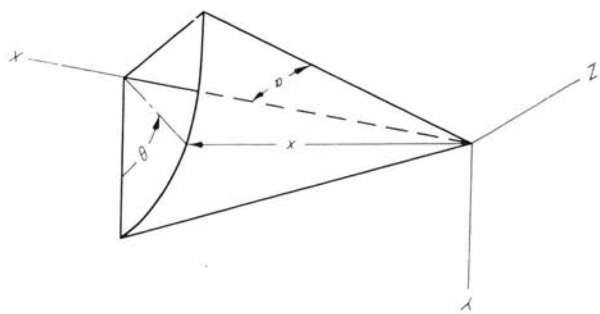

Figure 3

By Eqs. (4), (9), (11), and (13),

$$\left.\begin{aligned}
&E=1,\quad F=0,\quad G=(xs)^2,\quad A=1,\\
&B=xs,\quad 1/r_1=0,\quad r_2=xt=r,
\end{aligned}\right\}\qquad(53)$$

where $s=\sin\alpha$, $c=\cos\alpha$, $t=\tan\alpha$. With Eq. (53), Eqs. (35) and (38) become

$$\begin{aligned}
&P_1=u_x,\quad P_2=\frac{v_\theta}{xs},\\
&Q_1=0,\quad Q_2=\frac{u}{x}-\frac{w}{xt},\\
&R_1=w_{xx},\quad R_2=\frac{w_{\theta\theta}}{x^2s^2}+\frac{w_x}{x},\\
&J=v_x-\frac{v}{x},\quad K=\frac{u_\theta}{xs},\\
&L=\frac{v_x}{xt}-\frac{v}{x^2t}+\frac{w_{x\theta}}{xs}-\frac{w_\theta}{x^2s},
\end{aligned}\qquad(54)$$

$$\left.\begin{aligned} M &= \frac{w_{x\theta}}{xs} - \frac{w_\theta}{x^2 s}, \quad C_1 = 0, \\ C_2 &= 1 - \frac{xt}{h}\, ln\left(\frac{1 + h/2xt}{1 - h/2xt}\right) = C. \end{aligned}\right\} \tag{54}$$

Substituting Eq. (54) into Eq. (37), we obtain tensions, shears, and moments as follows:

$$\left.\begin{aligned}
N_x &= h\Big\{b_{11}\Big[u_x + \frac{h^2}{12xt}\, w_{xx}\Big] + b_{12}\Big[\frac{u}{x} + \frac{v_\theta}{xs} - \frac{w}{xt}\Big] + b_{13}\Big[v_x - \frac{v}{x} + \frac{u_\theta}{xs} \\
&\quad + \frac{h^2}{12x^2t^2}\left(v_x - \frac{v}{x} + \frac{w_{x\theta}}{c} - \frac{w_\theta}{xc}\right)\Big]\Big\} - \int_{-h/2}^{h/2} c_1 T\,(1 - z/xt)\, dz, \\
N_\theta &= h\Big\{b_{12}\, u_x + b_{22}\Big[\frac{v_\theta}{xs} + (1 - C)\left(\frac{u}{x} - \frac{w}{xt}\right) + Ct\left(w_x + \frac{w_{\theta\theta}}{xs^2}\right)\Big] \\
&\quad + b_{23}\Big[v_x - \frac{v}{x} + \frac{u_\theta}{xs}\,(1 - C) + \frac{C}{c}\left(w_{x\theta} - \frac{w_\theta}{x}\right)\Big]\Big\} - \int_{-h/2}^{h/2} c_2 T\, dz, \\
N_{x\theta} &= h\Big\{b_{13}\Big[u_x + \frac{h^2}{12xt}\, w_{xx}\Big] + b_{23}\Big[\frac{v_\theta}{xs} + \frac{u}{x} - \frac{w}{xt}\Big] + b_{33}\Big[\frac{u_\theta}{xs} + v_x - \frac{v}{x} \\
&\quad + \frac{h^2}{12x^2t^2}\left(v_x - \frac{v}{x} + \frac{w_{x\theta}}{c} - \frac{w_\theta}{xc}\right)\Big]\Big\} - \int_{-h/2}^{h/2} c_3 T\,(1 - z/xt)\, dz, \\
N_{\theta x} &= h\Big\{b_{13}\, u_x + b_{23}\Big[\frac{v_\theta}{xs} + (1 - C)\left(\frac{u}{x} - \frac{w}{xt}\right) + tC\left(w_x + \frac{w_{\theta\theta}}{xs^2}\right)\Big] \\
&\quad + b_{33}\Big[v_x - \frac{v}{x} + (1 - C)\,\frac{u_\theta}{xs} + \frac{C}{c}\left(w_{x\theta} - \frac{w_\theta}{x}\right)\Big]\Big\} - \int_{-h/2}^{h/2} c_3 T\, dz, \\
M_x &= \frac{h^3}{12xt}\Big\{b_{11}\,[u_x + xtw_{xx}] + b_{12}\Big[\frac{v_\theta}{xs} + t\left(w_x + \frac{w_{\theta\theta}}{xs^2}\right)\Big] \\
&\quad + 2b_{13}\Big[v_x - \frac{v}{x} + \frac{1}{c}\left(w_{x\theta} - \frac{w_\theta}{x}\right)\Big]\Big\} - \int_{-h/2}^{h/2} c_1 T\, z\,(1 - z/xt)\, dz, \\
M_\theta &= -\frac{h^3}{12}\Big\{b_{22}\,\frac{12tC}{h^2}\Big[u - \frac{w}{t} - \frac{w_{\theta\theta}}{cs} - xtw_x\Big] + b_{12}\, w_{xx} \\
&\quad + b_{23}\Big[\frac{1}{xt}\left(v_x - \frac{v}{x} + \frac{w_{x\theta}}{c} - \frac{w_\theta}{xc}\right) + \frac{12Ct}{h^2}\left(\frac{u_\theta}{s} - \frac{xw_{x\theta}}{c} + \frac{w_\theta}{c}\right)\Big]\Big\} - \int_{-h/2}^{h/2} c_2 T\, z\, dz, \\
M_{x\theta} &= -\frac{h^3}{12xt}\Big\{b_{13}\,[u_x + xtw_{xx}] + b_{23}\Big[\frac{v_\theta}{xs} + t\left(w_x + \frac{w_{\theta\theta}}{xs^2}\right)\Big] \\
&\quad + 2b_{33}\Big[v_x - \frac{v}{x} + t\left(\frac{w_{x\theta}}{s} - \frac{w_\theta}{xs}\right)\Big]\Big\} - \int_{-h/2}^{h/2} c_3 T\, z\,(1 - z/xt)\, dz, \\
M_{\theta x} &= -\frac{h^3}{12}\Big\{b_{13}\, w_{xx} + b_{23}\Big[u - \frac{w}{t} - \frac{w_{\theta\theta}}{sc} - txw_x\Big]\frac{12tC}{h^2} \\
&\quad + b_{33}\Big[\frac{1}{xt}\left(v_x - \frac{v}{x} + \frac{w_{x\theta}}{c} - \frac{w_\theta}{xc}\right) + \frac{12Ct}{h^2}\left(\frac{u_\theta}{s} - \frac{xw_{x\theta}}{c} + \frac{w_\theta}{c}\right)\Big]\Big\} - \int_{-h/2}^{h/2} c_3 T\, z\, dz.
\end{aligned}\right\} \tag{55}$$

With Eq. (53), Eqs. (26) through (31) become

$$\left.\begin{aligned}
&\frac{\partial N_x}{\partial x} + \frac{N_x}{x} + \frac{1}{xs}\frac{\partial N_{\theta x}}{\partial \theta} - \frac{N_\theta}{x} + P_x = 0 \\
&\frac{\partial N_{x\theta}}{\partial x} + \frac{N_{x\theta} + N_{\theta x}}{x} + \frac{1}{xs}\frac{\partial N_\theta}{\partial \theta} - \frac{Q_\theta}{xt} + P_\theta = 0 \\
&\frac{\partial Q_x}{\partial x} + \frac{Q_x}{x} + \frac{1}{xs}\frac{\partial Q_\theta}{\partial \theta} + \frac{N_\theta}{xt} + P_z = 0
\end{aligned}\right\} \tag{56}$$

$$\left.\begin{aligned}
&\frac{\partial M_x}{\partial x}+\frac{M_x}{x}+\frac{1}{xs}\frac{\partial M_{\theta x}}{\partial \theta}-\frac{M_\theta}{x}-Q_x=0\\
&\frac{\partial M_{x\theta}}{\partial x}+\frac{M_{x\theta}}{x}+\frac{1}{xs}\frac{\partial M_\theta}{\partial \theta}+\frac{M_{\theta x}}{x}-Q_\theta=0\\
&N_{\theta x}-N_{x\theta}=\frac{M_{\theta x}}{xt}.
\end{aligned}\right\} \quad (56)$$

In Eq. (56), the terms R_x and R_y have been discarded. By Eqs. (32) and (53), the moment equilibrium equation is

$$\begin{aligned}
&\frac{\partial^2 M_x}{\partial x^2}+\frac{2}{x}\frac{\partial M_x}{\partial x}+\frac{1}{xs}\frac{\partial^2(M_{\theta x}+M_{x\theta})}{\partial x \partial \theta}\\
&\quad -\frac{1}{x}\frac{\partial M_\theta}{\partial x}+\frac{1}{x^2 s}\frac{\partial(M_{x\theta}+M_{\theta x})}{\partial \theta}\\
&\quad +\frac{1}{x^2 s^2}\frac{\partial^2 M_\theta}{\partial \theta^2}+\frac{N_\theta}{xt}+P_z=0. \quad (57)
\end{aligned}$$

By Eqs. (34) and (54), the strains are

$$\left.\begin{aligned}
\epsilon_x&=u_x-z\,w_{xx}\\
\epsilon_\theta&=\frac{v_\theta}{xs}+\frac{u-w/t}{x-z/t}-\frac{z}{x-z/t}\left(w_x+\frac{w_{\theta\theta}}{xs^2}\right)\\
\gamma_{x\theta}&=v_x-\frac{v}{x}+\frac{u_\theta/s}{x-z/t}-\frac{z}{xt}\left(v_x-\frac{v}{x}\right.\\
&\quad\left.+\frac{w_{x\theta}}{c}-\frac{w_\theta}{xc}\right)-\frac{z/s}{x-z/t}\left(w_{x\theta}-\frac{w_\theta}{x}\right).
\end{aligned}\right\} \quad (58)$$

The strain energy for the conical shell is obtained by substitution of Eq. (54) into Eqs. (43) and (44) as

$$\begin{aligned}
U&=\iint\frac{h}{2}\left\{b_{11}\,u_x^2+\frac{1}{x^2}\,b_{22}\left(\frac{v_\theta}{s}+u-\frac{w}{t}\right)^2\right.\\
&\quad+b_{33}\left(v_x-\frac{v}{x}+\frac{u_\theta}{xs}\right)^2+2b_{12}\frac{u_x}{x}\left(\frac{v_\theta}{s}\right.\\
&\quad\left.+u-\frac{w}{t}\right)+2b_{13}\,u_x\left(v_x-\frac{v}{x}+\frac{u_\theta}{xs}\right)\\
&\quad+2b_{23}\frac{1}{x}\left(u+\frac{v_\theta}{s}-\frac{w}{t}\right)\left(v_x-\frac{v}{x}\right.\\
&\quad\left.\left.+\frac{u_\theta}{xs}\right)\right\}xs\,dx\,d\theta+\iint\frac{h^3}{24}\left\{b_{11}\left[w_{xx}{}^2\right.\right.\\
&\quad\left.+\frac{2u_xw_{xx}}{xt}\right]+b_{22}\left(\frac{-12C}{h^2}\right)\left[\frac{u}{x}-\frac{w}{xt}\right.\\
&\quad\left.-t\left(w_x+\frac{w_{\theta\theta}}{xs^2}\right)\right]^2+b_{33}\frac{3}{x^2s^2}\left[\left(cv_x\right.\right.\\
&\quad\left.-\frac{cv}{x}+w_{x\theta}-\frac{w_\theta}{x}\right)^2-\frac{4C}{h^2}(u_\theta-xtw_{x\theta}\\
&\quad\left.+tw_\theta)^2\right]+2b_{12}\frac{w_{xx}}{x}\left[w_x+\frac{w_{\theta\theta}}{xs^2}+\frac{v_\theta}{xst}\right]\\
&\quad+2b_{13}\frac{1}{xt}\left(2w_{xx}+\frac{u_x}{xt}\right)\left(v_x-\frac{v}{x}\right.\\
&\quad\left.+\frac{w_{x\theta}}{c}-\frac{w_\theta}{xc}\right)+2b_{23}\frac{1}{x^2}\left[\frac{1}{xst^2}\left(v_x\right.\right.\\
&\quad\left.-\frac{v}{x}+\frac{w_{x\theta}}{c}-\frac{w_\theta}{xc}\right)\left(v_\theta+\frac{w_{\theta\theta}}{c}\right.\\
&\quad\left.+xs\,t\,w_x\right)-\frac{12C}{sh^2}\left(u-\frac{w}{t}-\frac{w_{\theta\theta}}{sc}\right.\\
&\quad\left.\left.\left.-xtw_x\right)(u_\theta-xtw_{x\theta}+tw_\theta)\right]\right\}xs\,dx\,d\theta\\
&\quad+\iint Ixs\,dx\,d\theta \quad (59)
\end{aligned}$$

where

$$\begin{aligned}
I&=-\int_{-h/2}^{h/2}\left\{\left(1-\frac{z}{xt}\right)\left(c_1u_x+c_2\frac{v_\theta}{xs}\right.\right.\\
&\quad\left.+c_3v_x-c_3\frac{v}{x}\right)+c_3\frac{u_\theta}{xs}+c_2\frac{u}{x}-c_2\frac{w}{xt}\\
&\quad-c_2\frac{z}{x}\left(w_x+\frac{w_{\theta\theta}}{xs^2}\right)-c_3\frac{z}{xs}\left(w_{x\theta}-\frac{w_\theta}{x}\right)\\
&\quad-z\,(1-z/xt)\left[c_1w_{xx}+c_3\left(v_x-\frac{v}{x}\right.\right.\\
&\quad\left.\left.\left.+\frac{w_{x\theta}}{c}-\frac{w_\theta}{xc}\right)\frac{1}{xt}\right]\right\}T\,dz. \quad (60)
\end{aligned}$$

C. SPHERICAL SHELL

The middle surface of a spherical shell with center at the origin is given by the equations

$$\left.\begin{aligned}
&X=R\sin x\cos y,\quad Y=R\sin x\sin y,\\
&Z=R\cos x
\end{aligned}\right\} \quad (61)$$

where R is the radius of the middle surface, x is the colatitude, and y is the longitude (see Fig. 4). By Eqs. (4), (9), (11), (13), and (61),

$$\left.\begin{aligned}
&E=R^2,\quad G=R^2\sin^2 x,\quad f=0,\\
&A=R,\quad B=R\sin x,\quad r_1=r_2=-R.
\end{aligned}\right\} \quad (62)$$

With Eq. (62), Eqs. (35) and (38) become

$$\begin{aligned}
&P_1=\frac{u_x}{R},\quad P_2=\frac{v_y}{Rs},\quad Q_1=\frac{w}{R},\\
&Q_2=\frac{uc+w}{R},\quad R_1=\frac{w_{xx}}{R^2},
\end{aligned} \quad (63)$$

$$\left.\begin{aligned} R_2 &= \frac{w_{yy}}{R^2 s^2} + \frac{w_x c - uc}{R^2}, \quad J = \frac{v_x - vc}{R}, \\ K &= \frac{u_y}{Rs}, \quad L = \frac{vc - v_x}{R^2} + \frac{w_{xy} - w_y c}{R^2 s}, \\ M &= \frac{w_{xy} - c w_y - u_y}{R^2 s}, \quad C_1 = C_2 = 0, \end{aligned}\right\} \tag{63}$$

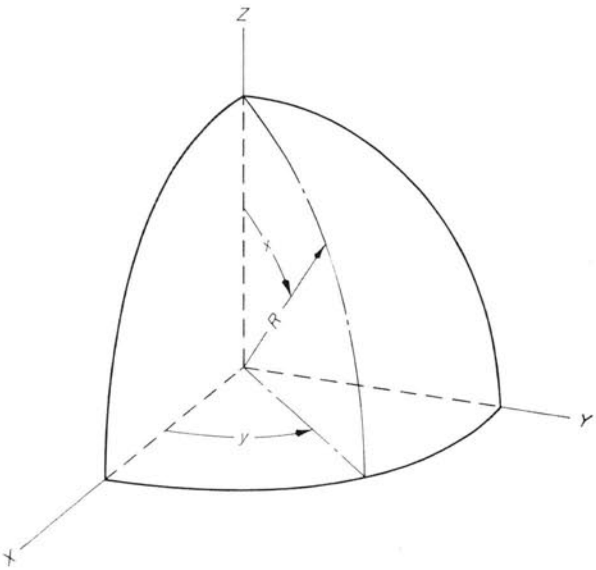

Figure 4

where $s = \sin x$, $c = \operatorname{ctn} x$. Substituting Eq. (63) into Eq. (37), we obtain tensions, shears, and moments as follows:

$$\left.\begin{aligned} N_x &= \frac{h}{R}\left\{b_{11}(u_x + w) + b_{12}\left(\frac{v_y}{s} + cu \right.\right. \\ &\quad \left.\left. + w\right) + b_{13}\left(\frac{u_y}{s} + v_x - cv\right)\right\} \\ &\quad - \int_{-h/2}^{h/2} c_1 T(1 + z/R)\, dz, \\ N_y &= \frac{h}{R}\left\{b_{12}(u_x + w) + b_{22}\left(\frac{v_y}{s} + cu \right.\right. \\ &\quad \left.\left. + w\right) + b_{23}\left(\frac{u_y}{s} + v_x - cv\right)\right\} \\ &\quad - \int_{-h/2}^{h/2} c_2 T(1 + z/R)\, dz, \\ N_{xy} &= N_{yx} = \frac{h}{R}\left\{b_{13}(u_x + w) + b_{23}\left(\frac{v_y}{s} \right.\right. \\ &\quad \left.\left. + cu + w\right) + b_{33}\left(\frac{u_y}{s} + v_x - cv\right)\right\} \end{aligned}\right| \tag{64}$$

$$\left.\begin{aligned} &\quad - \int_{-h/2}^{h/2} c_3 T(1 + z/R)\, dz, \\ M_x &= \frac{h^3}{12R^2}\left\{b_{11}(u_x - w_{xx}) + b_{12}\left(\frac{v_y}{s} \right.\right. \\ &\quad \left. + cu - cw_x - \frac{w_{yy}}{s^2}\right) + b_{13}\left(v_x \right. \\ &\quad \left.\left. - cv + \frac{2cw_y + u_y - 2w_{xy}}{s}\right)\right\} \\ &\quad - \int_{-h/2}^{h/2} c_1 Tz(1 + z/R)\, dz, \\ M_y &= \frac{h^3}{12R^2}\left\{b_{12}(u_x - w_{xx}) + b_{22}\left(\frac{v_y}{s} \right.\right. \\ &\quad \left. + cu - cw_x - \frac{w_{yy}}{s^2}\right) + b_{23}\left(v_x \right. \\ &\quad \left.\left. - cv + \frac{2cw_y + u_y - 2w_{xy}}{s}\right)\right\} \\ &\quad - \int_{-h/2}^{h/2} c_2 Tz(1 + z/R)\, dz, \\ M_{xy} &= M_{yx} = \frac{h^3}{12R^2}\left\{b_{13}(u_x - w_{xx}) \right. \\ &\quad + b_{23}\left(\frac{v_y}{s} + cu - cw_x - \frac{w_{yy}}{s^2}\right) \\ &\quad \left. + b_{33}\left(v_x - cv + \frac{2cw_y + u_y - 2w_{xy}}{s}\right)\right\} \\ &\quad - \int_{-h/2}^{h/2} c_3 Tz(1 + z/R)\, dz. \end{aligned}\right\} \tag{64}$$

Substitution of Eq. (62) into Eqs. (26) to (31) yields the following equilibrium equations:

$$\left.\begin{aligned} &\frac{\partial N_x}{\partial x} + \frac{1}{s}\cdot\frac{\partial N_{xy}}{\partial y} + c(N_x - N_y) \\ &\qquad + Q_x + RP_x = 0, \\ &\frac{\partial N_{xy}}{\partial x} + \frac{1}{s}\cdot\frac{\partial N_y}{\partial y} + 2cN_{xy} \\ &\qquad + Q_y + RP_y = 0, \\ &\frac{\partial Q_x}{\partial x} + \frac{1}{s}\cdot\frac{\partial Q_y}{\partial y} - (N_x + N_y) \\ &\qquad + cQ_x + RP_z = 0, \\ &\frac{\partial M_x}{\partial x} + \frac{1}{s}\cdot\frac{\partial M_{xy}}{\partial y} + c(M_x - M_y) \\ &\qquad - RQ_x + RR_y = 0, \end{aligned}\right\} \tag{65}$$

$$
\left.\begin{aligned}
&\frac{\partial M_{xy}}{\partial x} + \frac{1}{s}\cdot\frac{\partial M_y}{\partial y} + 2cM_{xy} \\
&\qquad - RQ_y - RR_x = 0.
\end{aligned}\right| \tag{65}
$$

By Eqs. (32) and (62), the moment equilibrium equation is

$$
\begin{aligned}
&\frac{\partial^2 M_x}{\partial x^2} + \frac{1}{s^2}\frac{\partial^2 M_y}{\partial y^2} + \frac{2}{s}\frac{\partial^2 M_{xy}}{\partial x \partial y} + c\frac{\partial}{\partial x}(2M_x \\
&\qquad - M_y) + 2\frac{c}{s}\frac{\partial M_{xy}}{\partial y} + (M_y - M_x) \\
&\qquad - R(N_x + N_y) + R^2 P_z = 0.
\end{aligned} \tag{66}
$$

Equations (34) and (63) yield

$$
\left.\begin{aligned}
\epsilon_x &= \frac{u_x}{R} + \frac{w}{R+z} - \frac{z w_{xx}}{R(R+z)}, \\
\epsilon_y &= \frac{uc}{R} + \frac{v_y}{Rs} + \frac{w}{R+z} \\
&\qquad - \frac{z}{(R+z)R}\left(\frac{w_{yy}}{s^2} + w_x c\right), \\
\gamma_{xy} &= \frac{v_x - vc}{R} + \frac{u_y}{Rs} \\
&\qquad - \frac{2z}{sR(R+z)}(w_{xy} - cw_y).
\end{aligned}\right\} \tag{67}
$$

The strain energy for the spherical shell is obtained from Eqs. (43), (44), and (63) as

$$
\begin{aligned}
U = \iint \frac{h}{2}\bigg\{ & b_{11}(w + u_x)^2 + b_{22}\left(\frac{v_y}{s}\right. \\
&\left. + w + cu\right)^2 + b_{33}\left(\frac{u_y}{s} + v_x - cv\right)^2 \\
& + 2b_{12}(u_x + w)\left(\frac{v_y}{s} + w + cu\right) \\
& + 2b_{13}(u_x + w)\left(\frac{u_y}{s} + v_x - cv\right) \\
& + 2b_{23}\left(\frac{v_y}{s} + w + cu\right)\left(\frac{u_y}{s} + v_x\right. \\
&\left. - cv\right)\bigg\} s\,dx\,dy + \iint \frac{h^3}{24R^2}\bigg\{ b_{11}(u_x - w_{xx})^2 \\
& + b_{22}\left(\frac{v_y}{s} + cu - cw_x - \frac{w_{yy}}{s^2}\right)^2 \\
& + b_{33}\left(\frac{u_y + 2cw_y - 2w_{xy}}{s} + v_x - cv\right)^2 \\
& + 2b_{12}(u_x - w_{xx})\left(\frac{v_y}{s} + cu - cw_x - \frac{w_{yy}}{s^2}\right) \\
& + 2b_{13}(u_x - w_{xx})\left(\frac{u_y + 2cw_y - 2w_{xy}}{s}\right. \\
&\left. + v_x - cv\right) + 2b_{23}\left(\frac{v_y}{s} + cu - cw_x\right. \\
&\left. - \frac{w_{yy}}{s^2}\right)\left(\frac{u_y + 2cw_y - 2w_{xy}}{s}\right. \\
&\left. + v_x - cv\right)\bigg\} s\,dx\,dy + \iint I\, R^2 S\,dx\,dy,
\end{aligned} \tag{68}
$$

where

$$
\begin{aligned}
I = -\int_{-h/2}^{h/2} \bigg\{ & \frac{(1 + z/R)^2}{R}\left[c_1 u_x + c_2\frac{v_y}{s}\right] \\
& + \frac{(1 + z/R)}{R}\left[c_1 w + c_2(w + cu) + c_3\left(v_x\right.\right. \\
&\left.\left. - cv + \frac{u_y}{s}\right)\right] - \frac{z(1 + z/R)}{R^2}\left[c_1 w_{xx}\right. \\
& + c_2\left(\frac{w_{yy}}{s^2} + cw_x - cu\right) + c_3\left(vc - v_x\right. \\
&\left.\left. + \frac{2w_{xy} - 2cw_y - u_y}{s}\right)\right]\bigg\} T\,dz.
\end{aligned} \tag{69}
$$

For isotropic shells the results derived in Sections A, B, and C reduce to those obtained by Langhaar and Boresi.[(3)(9)]

D. AXIALLY SYMMETRICAL ORTHOTROPIC SHELL OF REVOLUTION

The shell of revolution is shown in Fig. 5. The middle surface of the shell is described by $r = r(x)$

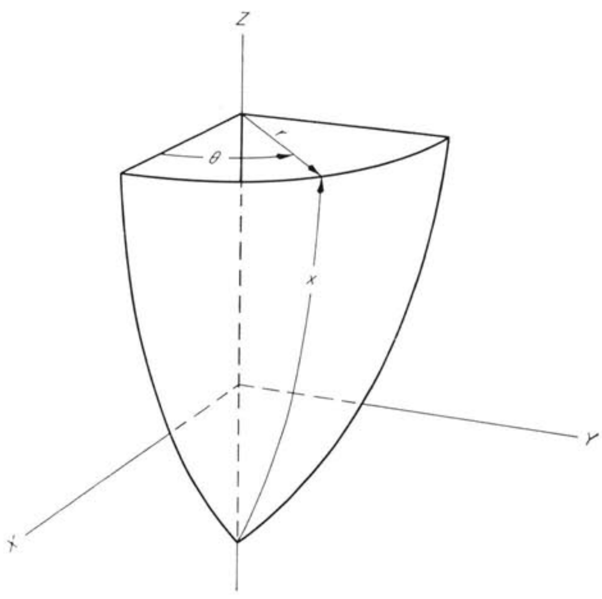

Figure 5

and $Z = Z(x)$. A point in the surface is located by the coordinates x and θ where x is arc length from the origin measured along a generator to the point and θ is the angle between the X-Z plane and the plane containing the point and the Z-axis. From Fig. 5,

$$X = r\cos\theta, \quad Y = r\sin\theta, \quad Z = Z. \tag{70}$$

Equations (4), (9), (11), (13), and (70) yield

$$\left.\begin{aligned} &E = 1,\ F = 0,\ G = r^2,\ A = 1,\ B = r, \\ &1/r_1 = r_x Z_{xx} - r_{xx} Z_x,\ 1/r_2 = Z_x/r. \end{aligned}\right\} \tag{71}$$

Consider an orthotropic shell of revolution which undergoes axially symmetrical deformation. Then

$$\begin{aligned} v = \frac{\partial}{\partial\theta} &= b_{13} = b_{23} = c_3 = N_{xy} = N_{yx} \\ &= M_{xy} = M_{yx} = Q_y = 0. \end{aligned} \tag{72}$$

With these restrictions, Eq. (35) becomes

$$\left.\begin{aligned} &P_1 = u_x, \quad P_2 = 0, \quad Q_1 = -\frac{w}{r_1}, \\ &Q_2 = \frac{ur_x}{r} - \frac{w}{r_2}, \quad R_1 = u\frac{\partial}{\partial x}\left(\frac{1}{r_1}\right) + w_{xx}, \\ &R_2 = \frac{1}{r}\left(\frac{ur_x}{r_1} + r_x w_x\right), \\ &J = K = L = M = 0. \end{aligned}\right\} \tag{73}$$

Substitution of Eq. (73) into Eq. (37) yields

$$\left.\begin{aligned} N_x &= h\Big\{b_{11}\Big(u_x - \frac{w}{r_1}(1 - C_1) \\ &\qquad + r_1 C_1\Big[u\frac{\partial}{\partial x}\Big(\frac{1}{r_1}\Big) + w_{xx}\Big]\Big) \\ &\qquad + b_{12}\Big(\frac{ur_x}{r} - \frac{w}{r_2}\Big)\Big\} \\ &\qquad - \int_{-h/2}^{h/2} c_1 T\,(1 - z/r_2)\,dz, \\ N_\theta &= h\Big\{b_{22}\Big[(1 - C_2)\Big(\frac{ur_x}{r} - \frac{w}{r_2}\Big) \\ &\qquad + r_2 C_2\Big(\frac{ur_x}{rr_1} + \frac{r_x w_x}{r}\Big)\Big] + b_{12}\Big[u_x \\ &\qquad - \frac{w}{r_1}\Big]\Big\} - \int_{-h/2}^{h/2} c_2 T\,(1 - z/r_1)\,dz, \\ M_x &= -\frac{h^3}{12 r_2}\Big\{b_{11}\Big(\Big[u_x - \frac{12 C_1}{h^2}\,wr_2\Big] \\ &\qquad + r_1\Big[u\frac{\partial}{\partial x}\Big(\frac{1}{r_1}\Big) + w_{xx}\Big]\Big[1 - \frac{12 r_1 r_2 C_1}{h^2}\Big]\Big) \\ &\qquad + b_{12} r_2\Big(\frac{ur_x}{rr_1} + \frac{r_x w_x}{r}\Big)\Big\} \\ &\qquad - \int_{-h/2}^{h/2} c_1 Tz\,(1 - z/r_2)\,dz, \\ M_\theta &= -\frac{h^3}{12 r_1}\Big\{b_{22}\Big[\frac{12}{h^2}\,C_2 r_1 r_2\Big(\frac{ur_x}{r} - \frac{w}{r_2}\Big) \\ &\qquad + r_2/r\Big(\frac{ur_x}{r_1} + r_x w_x\Big)\Big(1 - \frac{12 r_1 r_2 C_2}{h^2}\Big)\Big] \\ &\qquad + b_{12}\Big[u_x + r_1 u\frac{\partial}{\partial x}\Big(\frac{1}{r_1}\Big) + r_1 w_{xx}\Big]\Big\} \\ &\qquad - \int_{-h/2}^{h/2} c_2 Tz\,(1 - z/r_1)\,dz. \end{aligned}\right\} \tag{74}$$

With Eq. (71), the equilibrium equations (Eqs. (26) to (31)) become

$$\left.\begin{aligned} &\frac{\partial}{\partial x}(rN_x) - r_x N_\theta - \frac{r}{r_1}Q_x + rP_x = 0 \\ &\frac{\partial}{\partial x}(rQ_x) + \frac{r}{r_1}N_x + \frac{r}{r_2}N_\theta + rP_z = 0 \\ &\frac{\partial}{\partial x}(rM_x) - M_\theta r_x - rQ_x + rR_\theta = 0. \end{aligned}\right\} \tag{75}$$

Substitution of Eq. (71) into Eq. (32) yields

$$\begin{aligned} r\frac{\partial^2 M_x}{\partial x^2} &+ r_x\frac{\partial}{\partial x}(2M_x - M_\theta) + r_{xx}(M_x - M_\theta) \\ &+ \frac{r}{r_1}N_x + \frac{r}{r_2}N_\theta + rP_z = 0. \end{aligned} \tag{76}$$

The strains, given by Eqs. (34) and (73), are

$$\left.\begin{aligned} \epsilon_x &= u_x - \frac{w}{r_1 - z} \\ &\qquad - \Big[u\frac{\partial}{\partial x}\Big(\frac{1}{r_1}\Big) + w_{xx}\Big]\frac{z}{1 - z/r_1}, \\ \epsilon_\theta &= \Big(\frac{ur_x}{r} - \frac{w}{r_2}\Big)\frac{1}{1 - z/r_2} \\ &\qquad - \frac{1}{r}\Big(\frac{ur_x}{r_1} + r_x w_x\Big)\frac{z}{1 - z/r_2}, \\ \gamma_{x\theta} &= 0. \end{aligned}\right\} \tag{77}$$

By Eqs. (43) and (73), the strain energy of the shell is

$$\begin{aligned} U = \iint h\Big\{&\frac{b_{11}}{2}\Big[u_x - \frac{w}{r_1}\Big]^2 + \frac{b_{22}}{2}\Big[\frac{ur_x}{r} - \frac{w}{r_2}\Big]^2 \\ &+ b_{12}\Big[u_x - \frac{w}{r_1}\Big]\Big[\frac{ur_x}{r} - \frac{w}{r_2}\Big]\Big\}\,rdxd\theta \end{aligned}$$

$$
\begin{aligned}
&+\iint \frac{h^3}{12}\left\{\frac{b_{11}}{2}\left(\frac{\left[u_x+r_1u\frac{\partial}{\partial x}\left(\frac{1}{r_1}\right)+r_1w_{xx}\right]^2}{r_1r_2}\right.\right.\\
&-\left.\frac{12C_1}{h^2}\left[-\frac{w}{r_1}-ur_1\frac{\partial}{\partial x}\left(\frac{1}{r_1}\right)-r_1w_{xx}\right]^2\right)\\
&+\frac{b_{22}}{2}\left[\frac{r_2}{r_1}\left(\frac{ur_x}{rr_1}+\frac{r_xw_x}{r}\right)^2\right.\\
&-\left.\frac{12C_2}{h^2}\left(\frac{ur_x}{r}-\frac{w}{r_2}-\frac{r_2ur_x}{rr_1}-\frac{r_2r_x}{r}\,w_x\right)^2\right]\\
&+b_{12}\left(\frac{\left[u_x+r_1u\frac{\partial}{\partial x}\left(\frac{1}{r_1}\right)+r_1w_{xx}\right]}{r_1}\right.\\
&\left.\left.\left[\frac{ur_x}{rr_1}+\frac{r_xw_x}{r}\right]\right)\right\}rdxd\theta+\iint Irdxd\theta
\end{aligned}
\tag{78}
$$

where

$$
\begin{aligned}
I = &-\int_{-h/2}^{h/2}\left\{(1-z/r_1)\,(1-z/r_2)\,c_1u_x\right.\\
&+(1-z/r_1)\,c_2\left(\frac{ur_x}{r}-\frac{w}{r_2}\right)\\
&+(1-z/r_2)\left(-c_1\frac{w}{r_1}\right)-z\frac{c_2}{r}\,(1\\
&-z/r_1)\left(\frac{ur_x}{r_1}+r_xw_x\right)-z\,(1\\
&\left.-z/r_2)\,c_1\left[u\frac{\partial}{\partial x}\left(\frac{1}{r_1}\right)+w_{xx}\right]\right\}Tdz.
\end{aligned}
\tag{79}
$$

Equations (74) through (79) may be expressed in terms of displacement components V_1 and V_2 in the axial and radial directions, respectively, as

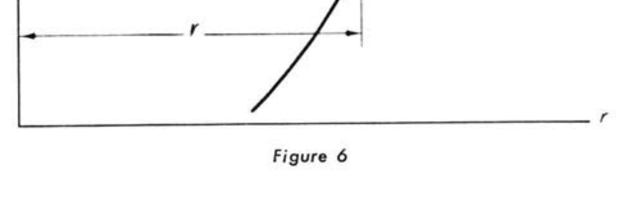

Figure 6

selected by Reissner.[10] Referring to Fig. 6, we obtain the relations for u and w in terms of V_1 and V_2 as follows:

$$
\left.\begin{aligned}
u &= V_1\sin\psi + V_2\cos\psi\\
w &= V_1\cos\psi - V_2\sin\psi
\end{aligned}\right\}
\tag{80}
$$

where $\psi\,(x)$ is the angle between the normal to the middle surface and the Z-axis. In addition, the radii of curvature may be expressed in the form

$$
\frac{1}{r_1}=\psi_x,\quad \frac{1}{r_2}=\frac{\sin\psi}{r}.
\tag{81}
$$

It is convenient to introduce the quantities

$$
\left.\begin{aligned}
\varphi &= (V_1)_x\cos\psi - (V_2)_x\sin\psi\\
\epsilon &= (V_1)_x\sin\psi + (V_2)_x\cos\psi
\end{aligned}\right\}
\tag{82}
$$

where φ is the rotation under deformation of the normal to the middle surface and ϵ is the strain of a generator on the middle surface.

With Eqs. (80), (81), and (82), the tractions become

$$
\left.\begin{aligned}
N_x &= h\left\{b_{11}\left[(1-C_1)\,\epsilon+\frac{C_1\varphi_x}{\psi_x}\right]+b_{12}\frac{V_2}{r}\right\}\\
&\quad-\int_{-h/2}^{h/2}c_1T\left(1-\frac{z\sin\psi}{r}\right)dz,\\
N_\theta &= h\left\{b_{22}\left[(1-C_2)\frac{V_2}{r}+\frac{C_2\varphi}{\tan\psi}\right]+b_{12}\,\epsilon\right\}\\
&\quad-\int_{-h/2}^{h/2}c_2T\,(1-z\psi_x)\,dz,\\
M_x &= \frac{-h^3}{12r}\left\{\frac{b_{11}}{\psi_x}\left[\frac{12C_1r}{h^2}\left(\epsilon-\frac{\varphi_x}{\psi_x}\right)\right.\right.\\
&\quad\left.\left.+\varphi_x\sin\psi\right]+b_{12}\,\varphi\cos\psi\right\}\\
&\quad-\int_{-h/2}^{h/2}c_1Tz\left(1-\frac{z\sin\psi}{r}\right)dz,\\
M_\theta &= -\frac{h^3}{12}\left\{b_{22}\left[\frac{12C_2}{h^2\sin\psi}\left(V_2-\frac{r\,\varphi}{\tan\psi}\right)\right.\right.\\
&\quad\left.\left.+\frac{\varphi\psi_x}{\tan\psi}\right]+b_{12}\,\varphi_x\right\}\\
&\quad-\int_{-h/2}^{h/2}c_2Tz\,(1-z\psi_x)\,dz.
\end{aligned}\right\}
\tag{83}
$$

Similarly, the equilibrium equations (Eqs. (75) and (76)) become

$$
\frac{\partial}{\partial x}\,(rN_x)-\cos\psi\,N_\theta-r\psi_xQ_x+rP_x=0,
\tag{84}
$$

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}(rQ_x) + r\psi_x N_x + \sin\psi\, N_\theta + rP_z = 0,\\
&\frac{\partial}{\partial x}(rM_x) - \cos\psi\, M_\theta - rQ_x + rR_\theta = 0,\\
&r\frac{\partial^2 M_x}{\partial x^2} + \cos\psi \frac{\partial}{\partial x}(2M_x - M_\theta)\\
&\qquad - \psi_x \sin\psi\,(M_x - M_\theta)\\
&\qquad + \psi_x r N_x + \sin\psi\, N_\theta + rP_z = 0.
\end{aligned}\right\} \quad (84)$$

The strains are obtained by substitution of Eqs. (80), (81), and (82) into Eq. (77). The results are

$$\left.\begin{aligned}
\epsilon_x &= \frac{\epsilon - z\varphi_x}{1 - z\psi_x},\\
\epsilon_\theta &= \frac{V_2 - z\varphi\cos\psi}{r - z\sin\psi},\\
\gamma_{x\theta} &= 0.
\end{aligned}\right\} \quad (85)$$

The strain energy expression (Eqs. (78) and (79)) may be written in terms of V_1 and V_2 in the form

$$\begin{aligned}
U = {} & \pi\int h\left\{b_{11}\,\epsilon^2 + b_{22}\left(\frac{V_2}{r}\right)^2 + 2b_{12}\frac{\epsilon V_2}{r}\right\} r\,dx\\
& + \pi\int \frac{h^3}{12}\left\{b_{11}\left[\frac{\psi_x \sin\psi}{r}\left(\frac{\varphi_x}{\psi_x}\right)^2\right.\right.\\
& \left. - \frac{12C_1}{h^2}\left(\frac{\varphi_x}{\psi_x} - \epsilon\right)^2\right] + b_{22}\left[\frac{\psi_x\cos\psi}{r\tan\psi}\varphi^2\right.\\
& \left. - \frac{12C_2}{h^2}\left(\frac{V_2}{r} - \frac{\varphi}{\tan\psi}\right)^2\right]\\
& \left. + 2b_{12}\left(\frac{\varphi_x\varphi\cos\psi}{r}\right)\right\} r\,dx + 2\pi\int I r\,dx
\end{aligned} \quad (86)$$

where

$$\begin{aligned}
I = {} & -\int_{-h/2}^{h/2}\left[c_1\left(1 - \frac{z\sin\psi}{r}\right)(\epsilon - z\varphi_x)\right.\\
& \left. + \frac{c_2(1 - z\psi_x)}{r}(V_2 - z\varphi\cos\psi)\right] T\,dz.
\end{aligned} \quad (87)$$

Solutions to problems of axially symmetrical shells of revolution may be obtained with either system of equations; i.e., equations (74) through (79) or equations (83) through (87). The choice will depend on the type of loading and the boundary conditions that the particular problem presents.

IV. SPECIAL PROBLEMS OF CYLINDRICAL SHELLS

A. AXIALLY SYMMETRICAL DEFORMATION OF AN ORTHOTROPIC CIRCULAR CYLINDER

Consider a circular cylinder, with radius a, oriented so that its axial coordinate x and its circumferential coordinate y coincide with the principal directions of orthotropy. Let the cylinder undergo axially symmetrical deformation. Then

$$\begin{aligned} v = \frac{\partial}{\partial y} &= b_{13} = b_{23} = c_3 = N_{xy} = N_{yx} \\ &= M_{xy} = M_{yx} = Q_y = 0. \end{aligned} \tag{88}$$

The equilibrium equations, determined by Eqs. (48), (49), and (88), are

$$\left.\begin{aligned} &\frac{\partial N_x}{\partial x} = 0, \quad \frac{\partial M_x}{\partial x} - Q_x = 0, \\ &\frac{\partial Q_x}{\partial x} + \frac{N_y}{a} + P_z = 0, \\ &\frac{\partial^2 M_x}{\partial x^2} + \frac{N_y}{a} + P_z = 0. \end{aligned}\right\} \tag{89}$$

By Eq. (47), the tractions are

$$\left.\begin{aligned} N_x &= h\left[b_{11}\left(u_x + \frac{h^2 w_{xx}}{12a}\right) - b_{12}\frac{w}{a}\right] \\ &\quad - \int_{-h/2}^{h/2} c_1 T\,(1 - z/a)\,dz, \\ N_y &= h\left[b_{22}\,(C-1)\,\frac{w}{a} + b_{12}\,u_x\right] \\ &\quad - \int_{-h/2}^{h/2} c_2 T\,dz, \\ M_x &= \frac{-h^3 b_{11}}{12a}\,(u_x + a w_{xx}) \\ &\quad - \int_{-h/2}^{h/2} c_1 T z\,(1 - z/a)\,dz, \\ M_y &= \frac{-h^3}{12a}\left(\frac{-12a\,C\,b_{22} w}{h^2} + b_{12} a w_{xx}\right) \\ &\quad - \int_{-h/2}^{h/2} c_2 T z\,dz. \end{aligned}\right\} \tag{90}$$

The strain energy for the axially symmetrical deformation of an orthotropic circular cylinder is obtained from Eqs. (51) and (52). It is

$$\begin{aligned} U &= \pi \int a h\left[b_{11} u_x^2 + b_{22}\left(\frac{w}{a}\right)^2 - 2b_{12}\frac{u_x w}{a}\right] dx \\ &\quad + \pi \int \frac{a h^3}{12}\left[b_{11}\left(w_{xx}^2 + \frac{2u_x w_{xx}}{a}\right)\right. \\ &\quad \left. - b_{22}\left(\frac{12C}{h^2 a^2}\right) w^2\right] dx + 2\pi \int a I\,dx \end{aligned} \tag{91}$$

where

$$\begin{aligned} I = -\int_{-h/2}^{h/2} &\left[c_1\,(1 - z/a)\,(u_x - z w_{xx})\right. \\ &\left. - c_2\frac{w}{a}\right] T\,dz. \end{aligned} \tag{92}$$

If the elastic constants b_{ij} are considered independent of x, the equilibrium equations may be expressed conveniently in terms of displacement components u and w. Substituting Eq. (90) into Eq. (89), we obtain

$$\left.\begin{aligned} &\frac{h^2}{12} w_{xxx} + a u_{xx} - \frac{b_{12}}{b_{11}} w_x \\ &\quad - \frac{a}{h b_{11}} \int_{-h/2}^{h/2} \left[\frac{\partial}{\partial x}(c_1 T)\right](1 - z/a)\,dz = 0, \\ &\frac{h^2}{12} w_{xxxx} - \frac{h b_{22}}{a^2 b_{11}}(C-1)\,w + \frac{h^3}{12a} u_{xxx} \\ &\quad - \frac{h b_{12}}{a b_{11}} u_x - \frac{P_z}{b_{11}} + \frac{1}{a b_{11}} \int_{-h/2}^{h/2} c_2 T dz \\ &\quad + \frac{1}{b_{11}} \int_{-h/2}^{h/2} \left[\frac{\partial^2}{\partial x^2}(c_1 T)\right] z\,(1 - z/a)\,dz = 0. \end{aligned}\right\} \tag{93}$$

With the preceding equations, analyses of problems of axially symmetrical deformations of orthotropic circular cylinders may be obtained by two procedures. One may express the total potential energy of the shell as a function of u and w and then minimize this function. Alternatively, one may solve the equilibrium equations (Eq. (93)) subject to the boundary conditions of the particular problem. In the following articles, examples are presented which serve to illustrate the two procedures.

B. CIRCULAR CYLINDER SUBJECTED TO TEMPERATURE WHICH VARIES THROUGH THE THICKNESS

Let the cylinder be constrained at its ends so that there is no strain in the x direction (i.e., $u=0$). This implies that the material properties, the radial displacement component w, and the temperature distribution are independent of x. Hence, the strain energy is

$$U = 2\pi \int_{-l}^{l} \left[\frac{h(1-C)b_{22}}{2a} w^2 + w \int_{-h/2}^{h/2} c_2 T dz \right] dx. \quad \textbf{(94)}$$

Since the integrand of Eq. (94) is independent of x, the integration is readily performed. The result is

$$U = 4\pi l \left\{ \frac{hb_{22}(1-C)}{2a} w^2 + w \int_{-h/2}^{h/2} c_2 T dz \right\}, \quad \textbf{(95)}$$

where the length of the cylinder is $2l$.

If the cylinder is subjected to a pressure P_i on the inside and P_o on the outside, the potential energy of the external loads is

$$\Omega = 4\pi a l (P_i - P_o) w. \quad \textbf{(96)}$$

The total potential energy V of the system is the sum of the strain energy U and the potential energy Ω of the external loads. Hence, by Eqs. (95) and (96),

$$V = 4\pi l \left\{ \frac{hb_{22}(1-C)}{2a} w^2 + \left[\int_{-h/2}^{h/2} c_2 T dz + a(P_i - P_o) \right] w \right\}. \quad \textbf{(97)}$$

By the principle of stationary potential energy, a necessary condition for equilibrium to exist is

$$\frac{dV}{dw} = 0. \quad \textbf{(98)}$$

Therefore, by Eqs. (97) and (98), we obtain

$$w = -\frac{a}{hb_{22}(1-C)} \left[\int_{-h/2}^{h/2} c_2 T dz + a(P_i - P_o) \right]. \quad \textbf{(99)}$$

Substitution of Eq. (99) in Eqs. (36) and (50) yields the stress components as

$$\left.\begin{aligned} \sigma_x &= \frac{ab_{12}}{hb_{22}(1-C)(a-z)} \left[\int_{-h/2}^{h/2} c_2 T dz + a(P_i - P_o) \right] - c_1 T, \\ \sigma_y &= \frac{a}{h(1-C)(a-z)} \left[\int_{-h/2}^{h/2} c_2 T dz + a(P_i - P_o) \right] - c_2 T, \\ \tau_{xy} &= 0. \end{aligned}\right\} \quad \textbf{(100)}$$

Equations (99) and (100) express stress components σ_x, σ_y, τ_{xy} and deflection w for arbitrary temperature distribution through the shell thickness.

C. SEMI-INFINITE CYLINDER WITH END MOMENTS AND SHEARS

Consider a semi-infinite cylinder as shown in Fig. 7. From Eq. (89),

$$\frac{\partial N_x}{\partial x} = 0.$$

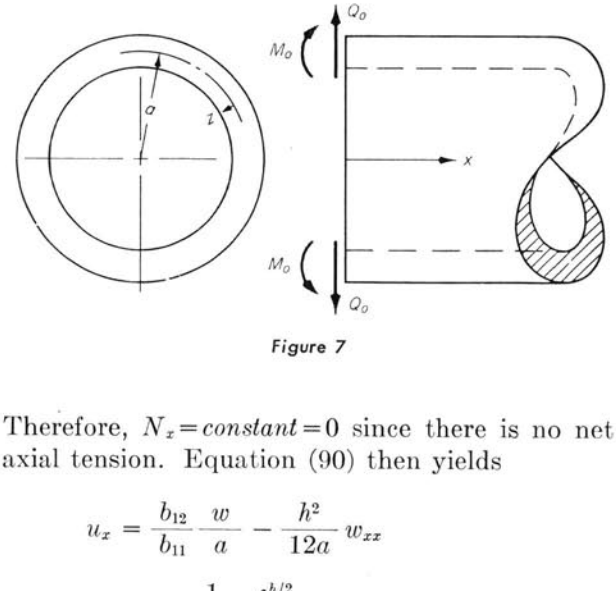

Figure 7

Therefore, $N_x = constant = 0$ since there is no net axial tension. Equation (90) then yields

$$u_x = \frac{b_{12}}{b_{11}} \frac{w}{a} - \frac{h^2}{12a} w_{xx} + \frac{1}{hb_{11}} \int_{-h/2}^{h/2} c_1 T (1 - z/a) dz. \quad \textbf{(101)}$$

Substitution of Eq. (101) into Eq. (93) yields the equilibrium equation for w:

$$w_{xxxx} + 4\alpha^2 w_{xx} + 4\beta^4 w = f(x). \quad \textbf{(102)}$$

Here,

$$\alpha^2 = \frac{b_{12}}{2a^2 b_{11}\delta}, \quad \textbf{(103)}$$

$$\left.\begin{aligned}
\beta^4 &= \frac{3}{a^2h^2\delta}\left\{\frac{b_{22}}{b_{11}}(1-C) - \frac{b_{12}^2}{b_{11}^2}\right\}, \\
f(x) &= \frac{12}{h^3 b_{11}\delta}\left\{\frac{1}{a}\int_{-h/2}^{h/2}\left[\frac{b_{12}}{b_{11}}c_1(1 - z/a) - c_2\right]T dz - \int_{-h/2}^{h/2}\left[\frac{d^2}{dx^2}(c_1 T)\right](1 - z/a)\left(z + \frac{h^2}{12a}\right)dz\right\}, \\
\delta &= 1 - \frac{h^2}{12a^2}.
\end{aligned}\right\} \quad (103)$$

The general solution of Eq. (102) is

$$\left.\begin{aligned}
w &= e^{-\sqrt{\beta^2-\alpha^2}\,x}\,[B_1 \cos\sqrt{\beta^2+\alpha^2}\,x + B_2 \sin\sqrt{\beta^2+\alpha^2}\,x] \\
&\quad + e^{\sqrt{\beta^2-\alpha^2}\,x}\,[B_3 \cos\sqrt{\beta^2+\alpha^2}\,x + B_4 \sin\sqrt{\beta^2+\alpha^2}\,x] + F(x)
\end{aligned}\right\} \quad (104)$$

where B_1, B_2, B_3, B_4 are arbitrary constants depending on the boundary conditions and $F(x)$ is the particular solution corresponding to the temperature function $f(x)$.

Since w must remain finite, $B_3 = B_4 = 0$. The constants B_1 and B_2 are determined by the boundary conditions at $x = 0$.

$$\text{When } x = 0, \quad M_x = M_o, \quad \frac{dM_x}{dx} = Q_o. \quad (105)$$

Since B_1 and B_2 depend on $F(x)$, consider the case $T = T_o e^{-x/a}$. Substitution of this function of T into Eq. (103) yields

$$f(x) = \frac{12}{ah^2 b_{11}\delta}\left(c_1 \frac{b_{12}}{b_{11}} - c_2\right)T_o e^{-x/a}. \quad (106)$$

Then, by Eq. (102),

$$F(x) = \gamma e^{-x/a} \quad (107)$$

where

$$\gamma = \frac{12a^3 T_o\left(c_1 \frac{b_{12}}{b_{11}} - c_2\right)}{b_{11}h^2\delta(1 + 4a^2\alpha^2 + 4a^4\beta^4)}. \quad (108)$$

Equation (104) then reduces to

$$w = e^{-\sqrt{\beta^2-\alpha^2}\,x}\,[B_1 \cos\sqrt{\beta^2+\alpha^2}\,x + B_2 \sin\sqrt{\beta^2+\alpha^2}\,x] + \gamma e^{-x/a}. \quad (109)$$

Substituting Eq. (109) into Eq. (90), eliminating u_x by Eq. (101), and applying the boundary conditions of Eq. (105), we obtain

$$\left.\begin{aligned}
B_1 &= \left\{\left[\frac{12M_o}{b_{11}h^3} + \frac{\gamma}{a^2}\left(\delta + \frac{b_{12}}{b_{11}}\right)\right]\left[2\delta(\beta^2 - 2\alpha^2) + \frac{b_{12}}{a^2 b_{11}}\right] - \left[\frac{\gamma}{a^3}\left(\delta + \frac{b_{12}}{b_{11}}\right) - \frac{12Q_o}{b_{11}h^3}\right]2\delta\sqrt{\beta^2-\alpha^2}\right\} \div \left\{\left[2\alpha^2\delta - \frac{b_{12}}{a^2b_{11}}\right]\left[2\delta(\beta^2 - 2\alpha^2) + \frac{b_{12}}{a^2 b_{11}}\right] - \left[2\delta(\beta^2 + 2\alpha^2) - \frac{b_{12}}{a^2b_{11}}\right]2\delta(\beta^2 - \alpha^2)\right\}, \\
B_2 &= \left\{\left[2\alpha^2\delta - \frac{b_{12}}{a^2 b_{11}}\right]\left[\frac{\gamma}{a^3}\left(\delta + \frac{b_{12}}{b_{11}}\right) - \frac{12Q_o}{b_{11}h^3}\right] - \left[2\delta(\beta^2 + 2\alpha^2) - \frac{b_{12}}{a^2b_{11}}\right]\left[\frac{\gamma}{a^2}\left(\delta + \frac{b_{12}}{b_{11}}\right) + \frac{12M_o}{b_{11}h^3}\right]\sqrt{\beta^2-\alpha^2}\right\} \div \left\{\left[2\alpha^2\delta - \frac{b_{12}}{a^2b_{11}}\right]\left[2\delta(\beta^2 - 2\alpha^2) + \frac{b_{12}}{a^2b_{11}}\right] - \left[2\delta(\beta^2 + 2\alpha^2) - \frac{b_{12}}{a^2b_{11}}\right]2\delta(\beta^2 - \alpha^2)\right\}\sqrt{\beta^2+\alpha^2}.
\end{aligned}\right\} \quad (110)$$

Substitution of Eqs. (103), (108), and (110) into Eq. (109) gives w as a function of x. Equations (101) and (109) yield u_x. Then Eqs. (36), (50), (101), and (109) give the stress components in the shell as functions of x and z.

D. NUMERICAL EXAMPLE

In this section, numerical values of deflections and stresses are computed with the results of Section C for five combinations of elastic coefficients b_{ij} and thermal coefficients c_i (see Table 1), for a cylinder with mean diameter of 20 inches, and with thickness of 1 inch. For isotropic material (Case 1),

$$\left.\begin{aligned} b_{11} &= b_{22} = \frac{E}{1-\nu^2}, \quad b_{12} = \frac{\nu E}{1-\nu^2}, \\ c_1 &= c_2 = \frac{Ek}{1-\nu} \end{aligned}\right\} \qquad (111)$$

where E is Young's modulus, ν is Poisson's ratio, and k is the coefficient of thermal expansion. In Case 2, the thermal coefficient in the longitudinal direction has been reduced by a factor of one-half as compared to Case 1. Similarly, in Case 3, the thermal coefficient in the circumferential direction has been reduced by a factor of one-half. In Case 4, the ratio of circumferential stiffness to longitudinal stiffness has been doubled; in Case 5, it has been reduced by a factor of one-half compared to Case 1. For brevity in the following discussion, the ratio of circumferential stiffness to longitudinal stiffness will be designated simply as the "stiffness ratio."

Two types of loading with constant temperature are considered: (a) a constant shear Q_o is applied at the free end of the cylinder (Fig. 7); (b) a constant moment M_o is applied at the free end of the cylinder (Fig. 7). Also, deflections and stresses are computed for a temperature distribution $T = T_o e^{-x/a}$, where T_o is a constant, x is the distance measured from the free end, and a is the mean radius of the cylinder.

The deflection w is computed by means of Eqs. (103), (108), (109), and (110). Numerical results are presented graphically in Figs. 8a, 8b, and 8c. Temperature effects are not included in Cases 1, 2, and 3 of Figs. 8a and 8b. In all cases, for end shear Q_o, the deflection w is a maximum at the free end (Fig. 8a). The stiffness ratio is doubled in Case 4. Then the deflection w attains a value of approximately 57% that attained in the isotropic case. When the stiffness ratio is reduced by one-half (Case 5), the deflection is 81% greater than in the isotropic case.

With end moment M_o, the deflection w is again largest at the free end (Fig. 8b). However, for end moment, the deflection w is less sensitive to changes in the stiffness ratio; in Case 4, w decreases to 69% of its value in Case 1, and in Case 5, it is increased by 48% of its value in the isotropic case.

Deflections due to temperature distribution $T = T_o e^{-x/a}$ are shown in Fig. 8c. As with end shear and end moment, the deflection w is a maximum at the free end. When the thermal coefficient in the longitudinal direction is decreased by a factor of one-half, compared to the isotropic

Table 1

Values of the Ratios of Elastic and Thermal Coefficients for the Five Cases of Article 14

Case No.	$\frac{b_{12}}{b_{11}}$	$\frac{b_{22}}{b_{11}}$	$\frac{c_1}{b_{11}}$ per °F	$\frac{c_2}{b_{11}}$ per °F
1	.3	1	10^{-5}	10^{-5}
2	.3	1	5×10^{-6}	10^{-5}
3	.3	1	10^{-5}	5×10^{-6}
4	.3	2	10^{-5}	10^{-5}
5	.3	.5	10^{-5}	10^{-5}

case, the deflection w increases by approximately 21% (Case 2, Fig. 8c); whereas, if the thermal coefficient in the circumferential direction is decreased by a factor of one-half, w decreases by approximately 71% (Case 3, Fig. 8c). Thus, for the assumed temperature distribution, the deflection w is more sensitive to changes in the thermal coefficient in the circumferential direction than in the longitudinal direction. Furthermore, a decrease of the thermal coefficient in the circumferential direction results in a decrease in the lateral deflection w; a decrease in the thermal coefficient in the longitudinal direction results in an increase in the lateral deflection. Hence, if lateral deflection due to temperature is to be kept small, a shell should be designed so that these two opposing effects cancel each other. Cases 4 and 5 of Fig. 8c illustrate the effect of the stiffness ratio. If the stiffness ratio is doubled, the deflection w is decreased to 52% of its value in the isotropic case. If the stiffness ratio is reduced by a factor of two, w is increased to 216% of its value in Case 1. Hence, for the type of temperature distribution considered, the deflection w is very sensitive to the stiffness ratio.

Numerical values of stress components σ_x and σ_y may be computed by means of Eqs. (36), (50), (101), and (109). In this example, σ_x and σ_y have been computed numerically for the free end ($x = 0$) for all three types of loading. The numerical values of σ_x and σ_y are illustrated graphically in Figs. 8d, 8e, and 8f.

For end shear Q_o, the longitudinal stress σ_x is zero at the free end (Fig. 8d). However, the circumferential stress σ_y varies through the thickness as shown; it attains a maximum value on the inside of the cylinder. When the stiffness ratio is doubled (Case 4), the maximum value of σ_y is increased by approximately 20% of its value in the isotropic case. If the stiffness ratio is decreased by a factor of two (Case 5), σ_y decreases approximately 18%. Consequently, σ_y is somewhat less sensitive to variations in the stiffness ratio than is deflection.

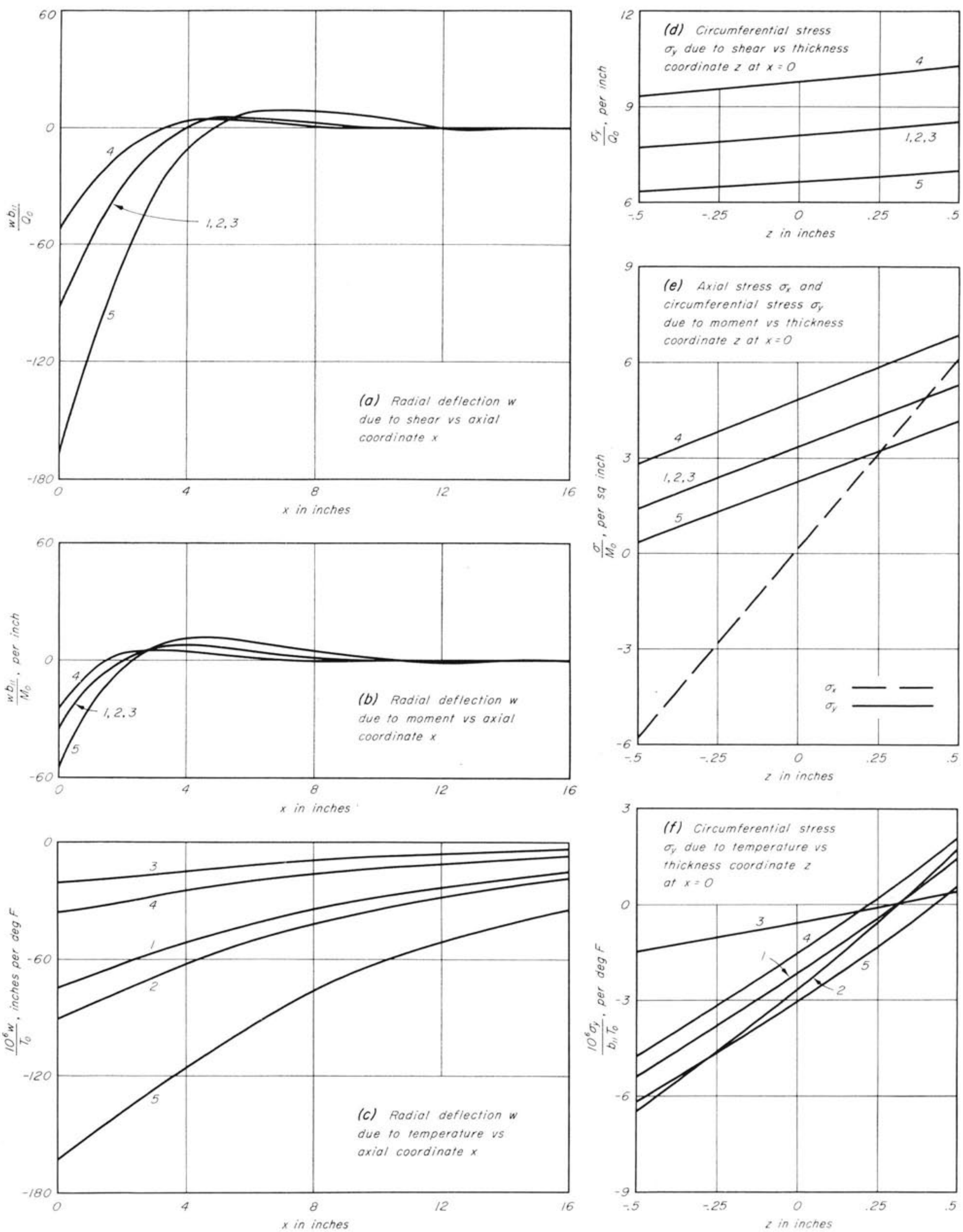
(a) Radial deflection w due to shear vs axial coordinate x
$\frac{w b_{11}}{Q_0}$
x in inches
4
1, 2, 3
5
(b) Radial deflection w due to moment vs axial coordinate x
$\frac{w b_{11}}{M_0}$, per inch
x in inches
(c) Radial deflection w due to temperature vs axial coordinate x
$\frac{10^6 w}{T_0}$, inches per deg F
x in inches
(d) Circumferential stress σ_y due to shear vs thickness coordinate z at x = 0
$\frac{\sigma_y}{Q_0}$, per inch
z in inches
(e) Axial stress σ_x and circumferential stress σ_y due to moment vs thickness coordinate z at x = 0
$\frac{\sigma}{M_0}$, per sq inch
σ_x
σ_y
z in inches
(f) Circumferential stress σ_y due to temperature vs thickness coordinate z at x = 0
$\frac{10^6 \sigma_y}{b_{11} T_0}$, per deg F
z in inches

Figure 8

For end moment M_o, the variations of stress components σ_x and σ_y are illustrated in Fig. 8e. The axial stress component σ_x is the same in all cases. The variation of the circumferential stress component σ_y is similar to that obtained for end shear Q_o.

For temperature distribution $T = T_o e^{-x/a}$, the circumferential stress component σ_y is shown in Fig. 8f, the axial stress component being zero at the free end. The largest compressive value of σ_y is attained in Case 2, its magnitude being approximately 21% larger than the value for the isotropic case. In Case 3, the maximum value of the compressive stress is approximately 72% less than for the isotropic case. If the stiffness ratio is doubled (Case 4), the maximum compressive stress is decreased by about 10%. However, the maximum tensile stress increases in comparison to the isotropic case (Fig. 8e). If the stiffness ratio is decreased by a factor of two (Case 5), the maximum compressive stress is increased approximately 16%, whereas the maximum tensile stress decreases.

In general, the maximum thermal stress is small. For example, for aluminum, b_{11} is approximately 10^7 lb/in.^{2} Then, for the value $T_o = 100$°F, the maximum compressive stress at the outer surface is approximately 540 lb/in.^{2}

V. DISCUSSION OF RESULTS

The geometrical preliminaries in Section I A are well known topics of the differential geometry of surfaces. Section I B explains the shell coordinates which are used throughout the remainder of the paper. The equilibrium and strain-displacement relations discussed in Sections II A and II B have been established previously in the theory of shells. The form in which they are presented here follows Langhaar and Boresi.[3]

In Section II C, a stress-strain-temperature relation for a linearly elastic anisotropic material is chosen. The equations for the tractions, shears, bending moments, and twisting moments in terms of the displacement components u, v, w of the middle surface are developed.

In Section II D, the strain energy of the shell in terms of the displacement components u, v, w is reduced to a surface integral by integration of the strain energy density through the thickness.

In Chapter III, Sections A, B, C, and D, the general theory is specialized for cylindrical shells, conical shells of circular cross section, spherical shells, and axially symmetrical orthotropic shells of revolution. The latter portion of Section D expresses the results for the shell of revolution in terms of axial and radial displacement components as selected by Reissner.[10]

As an illustration of the theory, two special problems for axially symmetrical orthotropic circular cylinders are analyzed in Chapter IV. Section B treats an infinite cylinder with temperature varying through the thickness. Section C treats a semi-infinite cylinder subjected to end moments and end shears and a temperature distribution which varies in the axial direction. The problems are solved by two different procedures. The first problem is solved by expressing the potential energy of the system in terms of the displacement w, and then applying the principle of stationary potential energy. The second problem is solved by expressing the equilibrium relation in terms of displacement components u and w. Then the resulting differential equations are solved for w.

For the second problem (Section IV D), numerical values for stresses and deflections are computed. The results are illustrated graphically in Figs. 8a, b, c, d, e, and f for the isotropic case and for four types of orthotropy.

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